Cell structures for the Yokonuma-Hecke algebra and the algebra of braids and ties

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ABSTRACT. We construct a faithful tensor representation for the Yokonuma-Hecke algebra $\mathcal{Y}_{r,n}$, and use it to give a concrete isomorphism between $\mathcal{Y}_{r,n}$ and Shoji's modified Ariki-Koike algebra. We give a cellular basis for $\mathcal{Y}_{r,n}$ and show that the Jucys-Murphy elements for $\mathcal{Y}_{r,n}$ are JM-elements in the abstract sense. Finally, we construct a cellular basis for the Aicardi-Juyumaya algebra of braids and ties.

Keywords: Yokonuma-Hecke algebra, Ariki-Koike algebra, cellular algebras.

MCS2010: 33D80.

1. INTRODUCTION

In the present paper, we study the representation theory of the Yokonuma-Hecke algebra $\mathcal{Y}_{r,n}$ in type A and of the related Aicardi-Juyumaya algebra $\mathcal{E}_n$ of braids and ties. In the past few years, quite a few papers have been dedicated to the study of both algebras.

The Yokonuma-Hecke algebra $\mathcal{Y}_{r,n}$ was first introduced in the sixties by Yokonuma [41] for general types but the recent activity on $\mathcal{Y}_{r,n}$ was initiated by Juyumaya who in [25] gave a new presentation of $\mathcal{Y}_{r,n}$. It is a deformation of the group algebra of the wreath product $C_r \wr \mathfrak{S}_n$ of the cyclic group C_r and the symmetric group $\mathfrak{S}_n$. On the other hand, it is quite different from the more familiar deformation of $C_r \wr \mathfrak{S}_n$, the Ariki-Koike algebra $\widetilde{\mathcal{H}}_{r,n}$. For example, the usual Iwahori-Hecke algebra $\mathcal{H}_n$ of type A appears canonically as a quotient of $\mathcal{Y}_{r,n}$, whereas it appears canonically as subalgebra of $\widetilde{\mathcal{H}}_{r,n}$.

Much of the impetus to the recent development on $\mathcal{Y}_{r,n}$ comes from knot theory. In the papers [9], [10], [24] and [26] a Markov trace on $\mathcal{Y}_{r,n}$ and its associated knot invariant Θ is studied.

The Aicardi-Juyumaya algebra $\mathcal{E}_n$ of braids and ties, along with its diagram calculus, was introduced in [1] and [23] via a presentation derived from the presentation of $\mathcal{Y}_{r,n}$. The algebra $\mathcal{E}_n$ is also related to knot theory. Indeed, Aicardi and Juyumaya constructed in [2] a Markov trace on $\mathcal{E}_n$, which gave rise to a three parameter knot invariant Δ. There seems to be no simple relation between Θ and Δ.

A main aim of our paper is to show that $\mathcal{Y}_{r,n}$ and $\mathcal{E}_n$ are cellular algebras in the sense of Graham and Lehrer, [14]. On the way we give a concrete isomorphism between $\mathcal{Y}_{r,n}$ and Shoji's modified Ariki-Koike algebra $\mathcal{H}_{r,n}$. This gives a new proof of a result of Lusztig [28] and Jacon-Poulain d'Andecy [21], showing that $\mathcal{Y}_{r,n}$ is in fact a sum of matrix algebra over Iwahori-Hecke algebras of type A.

For the parameter $q = 1$, it was shown in Banjo's work [4] that the algebra $\mathcal{E}_n$ is a special case of P. Martin's ramified partition algebras. Moreover, Marin showed in

*Supported in part by Beca Doctorado Nacional 2013-CONICYT 21130109

†Supported in part by FONDECYT grant 1121129

[29] that $\mathcal{E}_n$ for $q = 1$ is isomorphic to a sum of matrix algebras over a certain wreath product algebra, in the spirit of Lusztig's and Jacon-Poulain d'Andecy's Theorem. He raised the question whether this result could be proved for general parameters. As an application of our cellular basis for $\mathcal{E}_n$ we do obtain such a structure Theorem for $\mathcal{E}_n$, thus answering in the positive Marin's question.

Recently it was shown in [9] and [35] that the Yokonuma-Hecke algebra invariant Θ can be described via a formula involving the HOMFLYPT-polynomial and the linking number. In particular, when applied to classical knots, Θ and the HOMFLYPT-polynomial coincide (this was already known for some time). Given our results on $\mathcal{E}_n$ it would be interesting to investigate whether a similar result would hold for Δ.

Roughly our paper can be divided into three parts. The first part, sections 2 and 3, contains the construction of a faithful tensor space module $V^{\otimes n}$ for $\mathcal{Y}_{r,n}$. The construction of $V^{\otimes n}$ is a generalization of the $\mathcal{E}_n$-module structure on $V^{\otimes n}$ that was defined in [36] and it allows us to conclude that $\mathcal{E}_n$ is a subalgebra of $\mathcal{Y}_{r,n}$ for $r \geq n$, and for *any* specialization of the ground ring. The tensor space module $V^{\otimes n}$ is also related to the strange Ariki-Terasoma-Yamada action, [3] and [37], of the Ariki-Koike algebra on $V^{\otimes n}$, and thereby to the action of Shoji's modified Ariki-Koike algebra $\mathcal{H}_{r,n}$ on $V^{\otimes n}$, [39]. A speculating remark concerning this last point was made in [36], but the appearance of Vandermonde determinants in the proof of the faithfulness of the action of $\mathcal{Y}_{r,n}$ in $V^{\otimes n}$ makes the remark much more precise. The defining relations of the modified Ariki-Koike algebra also involve Vandermonde determinants and from this we obtain the proof of the isomorphism $\mathcal{Y}_{r,n} \cong \mathcal{H}_{r,n}$ by viewing both algebras as subalgebras of $\mathrm{End}(V^{\otimes n})$. Via this, we get a new proof of Lusztig's and Jacon-Poulain d'Andecy's isomorphism Theorem for $\mathcal{Y}_{r,n}$, since it is in fact equivalent to a similar isomorphism Theorem for $\mathcal{H}_{r,n}$, obtained independently by Sawada-Shoji and Hu-Stoll.

The second part of our paper, section 4 and 5, contains the proof that $\mathcal{Y}_{r,n}$ is a cellular algebra in the sense of Graham-Lehrer, via a concrete combinatorial construction of a cellular basis for it, generalizing Murphy's standard basis for the Iwahori-Hecke algebra of type A. The fact that $\mathcal{Y}_{r,n}$ is cellular could also have been deduced from the isomorphism $\mathcal{Y}_{r,n} \cong \mathcal{H}_{r,n}$ and from the fact that $\mathcal{H}_{r,n}$ is cellular, as was shown by Sawada and Shoji in [38]. Still, the usefulness of cellularity depends to a high degree on having a concrete cellular basis in which to perform calculations, rather than knowing the mere existence of such a basis, and our construction should be seen in this light.

Cellularity is a particularly strong language for the study of modular, that is non-semisimple representation theory, which occurs in our situation when the parameter q is specialized to a root of unity. But here our applications go in a different direction and depend on a nice compatibility property of our cellular basis with respect to a natural subalgebra of $\mathcal{Y}_{r,n}$. We get from this that the elements $m_{\mathfrak{s}\mathfrak{s}}$ of the cellular basis for $\mathcal{Y}_{r,n}$, given by one-column standard multitableaux $\mathfrak{s}$, correspond to certain idempotents that appear in Lusztig's presentation of $\mathcal{Y}_{r,n}$ in [27] and [28]. Using the faithfulness of the tensor space module $V^{\otimes n}$ for $\mathcal{Y}_{r,n}$ we get via this Lusztig's idempotent presentation of $\mathcal{Y}_{r,n}$. Thus the second part of the paper depends logically on the first part.

In section 5 we treat the Jucys-Murphy's elements for $\mathcal{Y}_{r,n}$. They were already introduced and studied by Chlouveraki and Poulain d'Andecy in [8], but here we show that they are JM-elements in the abstract sense defined by Mathas, with respect to the cell structure that we found.

The third part of our paper, section 6, contains the construction of a cellular basis for $\mathcal{E}_n$. This construction does not depend logically on the results of parts 1 and 2, but is still strongly motivated by them. The generic representation theory of $\mathcal{E}_n$ was already studied in [36] and was shown to be a blend of the symmetric group and the Hecke algebra representation theories and this is reflected in the cellular basis. The cellular basis is also here a variation of Murphy's standard basis but the details of the construction are substantially more involved than in the $\mathcal{Y}_{r,n}$-case.

As an application of our cellular basis we show that $\mathcal{E}_n$ is isomorphic to a direct sum of matrix algebras over certain wreath product algebras $\mathcal{H}_\alpha^{wr}$, depending on a partition α. An essential ingredient in the proof of this result is a compatibility property of our cellular basis for $\mathcal{E}_n$ with respect to these subalgebras. It appears to be a key feature of Murphy's standard basis and its generalizations that they carry compatibility properties of this kind, see for example [19], [12] and [13], and thus our work can be viewed as a manifestation of this phenomenon.

We thank the organizers of the Representation Theory Programme at the Institut Mittag-Leffler for the possibility to present this work in April 2015. We thank G. Lusztig for pointing out on that occasion that the isomorphism Theorem for $\mathcal{Y}_{r,n}$ is proved already in [28]. We thank J. Juyumaya for pointing out a missing condition in a first version of Theorem 14. Finally, we thank M. Chlouveraki and L. Poulain d'Andecy for sending us their comments and D. Plaza for many useful discussions on the topic of the paper.

2. Notation and Basic Concepts

In this section we set up the fundamental notation and introduce the objects we wish to investigate.

Throughout the paper we fix the rings $R := \mathbb{Z}[q, q^{-1}, \xi, r^{-1}, \Delta^{-1}]$ and $S := \mathbb{Z}[q, q^{-1}]$, where q is an indeterminate, r is a positive integer, $\xi := e^{2\pi i/r} \in \mathbb{C}$ and Δ is the Vandermonde determinant $\Delta := \prod_{0\le i<j\le r-1}(\xi^i - \xi^j)$.

We shall need the quantum integers $[m]_q$ defined for $m \in \mathbb{Z}$ by $[m]_q := \dfrac{q^{2m}-1}{q^2-1}$ if $q \neq 1$ and $[m]_q := m$ if $q = 1$.

Let $\mathfrak{S}_n$ be the symmetric group on n letters. We choose the convention that it acts on $\mathbf{n} := \{1, 2 \dots, n\}$ on the right. Let $\Sigma_n := \{s_1, \dots, s_{n-1}\}$ be the set of simple transpositions in $\mathfrak{S}_n$, that is $s_i = (i, i+1)$. Thus, $\mathfrak{S}_n$ is the Coxeter group on Σ_n subject to the relations

$$s_i s_j = s_j s_i \qquad \text{for } |i-j| > 1 \tag{1}$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad \text{for } i = 1, 2, \dots, n-2 \tag{2}$$
$$s_i^2 = 1 \qquad \text{for } i = 1, 2, \dots, n-1. \tag{3}$$

We let $\ell(\cdot)$ denote the usual length function on $\mathfrak{S}_n$.

Definition 1 Let n be a positive integer. The Yokonuma-Hecke algebra, denoted $\mathcal{Y}_{r,n} = \mathcal{Y}_{r,n}(q)$, is the associative R-algebra generated by the elements $g_1, \dots, g_{n-1}, t_1, \dots, t_n$,

subject to the following relations:

$$
\begin{aligned}
t_i^r &= 1 && \text{for all } i && (4)\\
t_i t_j &= t_j t_i && \text{for all } i,j && (5)\\
t_j g_i &= g_i t_{js_i} && \text{for all } i,j && (6)\\
g_i g_j &= g_j g_i && \text{for } |i-j|>1 && (7)\\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} && \text{for all } i=1,\ldots,n-2 && (8)
\end{aligned}
$$

together with the quadratic relation

$$g_i^2 = 1 + (q-q^{-1})e_i g_i \qquad \text{for all } i \tag{9}$$

where

$$e_i := \frac{1}{r}\sum_{s=0}^{r-1} t_i^s t_{i+1}^{-s}. \tag{10}$$

Note that since r is invertible in R, the element $e_i \in \mathcal{Y}_{r,n}(q)$ makes sense.

One checks that e_i is an idempotent and that g_i is invertible in $\mathcal{Y}_{r,n}(q)$ with inverse

$$g_i^{-1} = g_i + (q^{-1}-q)e_i. \tag{11}$$

The study of the representation theory $\mathcal{Y}_{r,n}(q)$ is one of the main themes of the present paper. $\mathcal{Y}_{r,n}(q)$ can be considered as a generalization of the usual Iwahori-Hecke algebra $\mathcal{H}_n = \mathcal{H}_n(q)$ of type A_{n-1} since $\mathcal{Y}_{1,n}(q) = \mathcal{H}_n(q)$. In general $\mathcal{H}_n(q)$ is a canonical quotient of $\mathcal{Y}_{r,n}(q)$ via the ideal generated by all the $t_i - 1$'s. On the other hand, as a consequence of the results of the present paper, $\mathcal{H}_n(q)$ also appears as a subalgebra of $\mathcal{Y}_{r,n}(q)$ although not canonically.

$\mathcal{Y}_{r,n}(q)$ was introduced by Yokonuma in the sixties as the endomorphism algebra of a module for the Chevalley group of type A_{n-1}, generalizing the usual Iwahori-Hecke algebra construction, see [41]. This also gave rise to a presentation for $\mathcal{Y}_{r,n}(q)$. A different presentation for $\mathcal{Y}_{r,n}(q)$, widely used in the literature, was found by Juyumaya. The presentation given above appeared first in [8] and differs slightly from Juyumaya's presentation. In Juyumaya's presentation another variable u is used and the quadratic relation (9) takes the form $\tilde{g}_i^2 = 1 + (u-1)e_i(\tilde{g}_i + 1)$. The relationship between the two presentations is given by $u = q^2$ and

$$\tilde{g}_i = g_i + (q-1)e_i g_i, \tag{12}$$

or equivalently $g_i = \tilde{g}_i + (q^{-1}-1)e_i\tilde{g}_i$, see eg. [9].

In this paper we shall be interested in the general, not necessarily semisimple, representation theory of $\mathcal{Y}_{r,n}(q)$ and shall therefore need base change of the ground ring. Let $\mathcal{K}$ be a commutative ring, with elements $q, \xi \in \mathcal{K}^\times$. Suppose moreover that ξ is an r'th root of unity and that r and $\prod_{0\le i<j\le r-1}(\xi^i - \xi^j)$ are invertible in $\mathcal{K}$ (for example $\mathcal{K}$ a field with $r, \xi \in \mathcal{K}^\times$ and ξ a primitive r'th root of unity). Then we can make $\mathcal{K}$ into an R-algebra by mapping $q \in R$ to $q \in \mathcal{K}$, and $\xi \in R$ to $\xi \in \mathcal{K}$. This gives rise to the specialized Yokonuma-Hecke algebra

$$\mathcal{Y}_{r,n}^{\mathcal{K}}(q) = \mathcal{Y}_{r,n}(q) \otimes_R \mathcal{K}.$$

Let $w \in \mathfrak{S}_n$ and suppose that $w = s_{i_1}s_{i_2}\cdots s_{i_m}$ is a reduced expression for w. Then by the relations the element $g_w := g_{i_1}g_{i_2}\cdots g_{i_m}$ does not depend on the choice of the

reduced expression for w. We use the convention that $g_1 := 1$. In [24] Juyumaya proved that the following set is an R-basis for $\mathcal{Y}_{r,n}(q)$

$$\mathcal{B}_{r,n} = \{t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n} g_w \mid w \in \mathfrak{S}_n,\ k_1, \ldots, k_n \in \mathbb{Z}/r\mathbb{Z}\}. \tag{13}$$

In particular, $\mathcal{Y}_{r,n}(q)$ is a free R-module of rank $r^n n!$. Similarly, $\mathcal{Y}_{r,n}^{\mathcal{K}}(q)$ is a free over $\mathcal{K}$ of rank $r^n n!$.

Let us introduce some useful elements of $\mathcal{Y}_{r,n}(q)$ (or $\mathcal{Y}_{r,n}^{\mathcal{K}}(q)$). For $1 \le i, j \le n$ we define e_{ij} by

$$e_{ij} := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_j^{-s}. \tag{14}$$

These e_{ij}'s are idempotents and $e_{ii} = 1$ and $e_{i,i+1} = e_i$. Moreover $e_{ij} = e_{ji}$ and it is easy to verify from (6) that

$$e_{ij} = g_i g_{i+1} \cdots g_{j-2} e_{j-1} g_{j-2}^{-1} \cdots g_{i+1}^{-1} g_i^{-1} \qquad \text{for } i < j. \tag{15}$$

From (4)-(6) one obtains that

$$t_i e_{ij} = t_j e_{ij} \qquad \text{for all } i, j \tag{16}$$

$$e_{ij} e_{kl} = e_{kl} e_{ij} \qquad \text{for all } i, j, k, l \tag{17}$$

$$e_{ij} g_k = g_k e_{i s_k, j s_k} \text{ for all } i, j \text{ and } k = 1, \ldots, n-1. \tag{18}$$

For any nonempty subset $I \subset \mathbf{n}$ we extend the definition of e_{ij} to e_I by setting

$$E_I := \prod_{i,j \in I, i<j} e_{ij} \tag{19}$$

where we use the convention that $E_I := 1$ if $|I| = 1$.

We need a further generalization of this. Recall that a set of subsets $A = \{I_1, I_2, \ldots I_k\}$ of $\mathbf{n}$ is called a *set partition* of $\mathbf{n}$ if the I_j's are nonempty, disjoint and have union $\mathbf{n}$. We refer to the I_i's as the *blocks* of A. The set of all set partitions of $\mathbf{n}$ is denoted $\mathcal{SP}_n$. There is a natural poset structure on $\mathcal{SP}_n$ defined as follows. Suppose that $A = \{I_1, I_2, \ldots, I_k\} \in \mathcal{SP}_n$ and $B = \{J_1, I_2, \ldots, J_l\} \in \mathcal{SP}_n$. Then we say that $A \subseteq B$ if each J_j is a union of some of the I_i's.

For any set partition $A = \{I_1, I_2, \ldots, I_k\} \in \mathcal{SP}_n$ we define

$$E_A := \prod_j E_{I_j}. \tag{20}$$

Extending the right action of $\mathfrak{S}_n$ on $\mathbf{n}$ to a right action on $\mathcal{SP}_n$ via $Aw := \{I_1 w, \ldots, I_k w\} \in \mathcal{SP}_n$ for $w \in \mathfrak{S}_n$, we have the following Lemma.

Lemma 2 For $A \in \mathcal{SP}_n$ and $w \in \mathfrak{S}_n$ as above, we have that

$$E_A g_w = g_w E_{Aw}.$$

In particular, if w leaves invariant every block of A, or more generally permutes certain of the blocks of A (of the same size), then E_A and g_w commute.

PROOF. This is immediate from (18) and the definitions. □

As mentioned above, the specialized Yokonuma-Hecke algebra $\mathcal{Y}_{r,n}^{\mathcal{K}}(q)$ only exists if r is a unit in $\mathcal{K}$. The algebra of braids and ties $\mathcal{E}_n(q)$, introduced by Aicardi and Juyumaya, is an algebra related to $\mathcal{Y}_{r,n}(q)$ that exists for any ground ring. It has a diagram calculus consisting of braids that may be decorated with socalled ties, which explains its name, see [1]. Here we only give its definition in terms of generators and relations.

Definition 3 Let n be a positive integer. The algebra of braids and ties, $\mathcal{E}_n = \mathcal{E}_n(q)$, is the associative S-algebra generated by the elements $g_1, \dots, g_{n-1}, e_1, \dots, e_{n-1}$, subject to the following relations:

$$
\begin{aligned}
g_i g_j &= g_j g_i && \text{for } |i-j| > 1 && (21)\\
g_i e_i &= e_i g_i && \text{for all } i && (22)\\
g_i g_j g_i &= g_j g_i g_j && \text{for } |i-j| = 1 && (23)\\
e_i g_j g_i &= g_j g_i e_j && \text{for } |i-j| = 1 && (24)\\
e_i e_j g_j &= e_i g_j e_i = g_j e_i e_j && \text{for } |i-j| = 1 && (25)\\
e_i e_j &= e_j e_i && \text{for all } i, j && (26)\\
g_i e_j &= e_j g_i && \text{for } |i-j| > 1 && (27)\\
e_i^2 &= e_i && \text{for all } i && (28)\\
g_i^2 &= 1 + (q - q^{-1}) e_i g_i && \text{for all } i. && (29)
\end{aligned}
$$

Once again, this differs slightly from the presentation normally used for $\mathcal{E}_n(q)$, for example in [36], where the variable u is used and the quadratic relation takes the form $\tilde{g}_i^2 = 1 + (u-1) e_i (\tilde{g}_i + 1)$. And once again, to change between the two presentations one uses $u = q^2$ and

$$g_i = \tilde{g}_i + (q^{-1} - 1) e_i \tilde{g}_i \tag{30}$$

For any commutative ring $\mathcal{K}$ containing the invertible element q, we define the specialized algebra $\mathcal{E}_n^{\mathcal{K}}(q)$ via $\mathcal{E}_n^{\mathcal{K}}(q) := \mathcal{E}_n(q) \otimes_S \mathcal{K}$ where $\mathcal{K}$ is made into an S-algebra by mapping $q \in S$ to $q \in \mathcal{K}$.

Lemma 4 Let $\mathcal{K}$ be a commutative ring containing invertible elements r, ξ, Δ as above. Then there is a homomorphim $\varphi : \mathcal{E}_n^{\mathcal{K}}(q) \to \mathcal{Y}_{r,n}^{\mathcal{K}}(q)$ of $\mathcal{K}$-algebras induced by $\varphi(g_i) := g_i$ and $\varphi(e_i) := e_i$.

PROOF. This is immediate from the relations. We shall later on show that φ is an embedding if $r \geq n$. $\square$

Let $\mathbb{N}^0$ denote the nonnegative integers. We next recall the combinatorics of Young diagrams and tableaux. A *composition* $\mu = (\mu_1, \mu_2, \dots, \mu_l)$ of $n \in \mathbb{N}^0$ is a finite sequence in $\mathbb{N}^0$ with sum n, The μ_i's are called the parts of μ. A *partition* of n is a composition whose parts are non-increasing. We write $\mu \models n$ and $\lambda \vdash n$ if μ is a composition of n and λ is a partition of n. In these cases we set $|\mu| := n$ and $|\lambda| := n$ and define the length of μ or λ as the number of parts of μ or λ. We denote by $\mathcal{C}omp_n$ the set of compositions of n and by $\mathcal{P}ar_n$ the set of partitions of n. The *Young diagram* of a composition μ is the subset

$$[\mu] = \{(i, j) \mid 1 \leq j \leq \mu_i \text{ and } i \geq 1\}$$

of $\mathbb{N}^0 \times \mathbb{N}^0$. The elements of $[\mu]$ are called the *nodes* of μ. We represent $[\mu]$ as an array of boxes in the plane, identifying each node with a box. For example, if $\mu = (3, 2, 4)$ then

$$[\mu] = \ydiagram{3,2,4}\,.$$

For $\mu \models n$ we define a μ-*tableau* as a bijection $\mathfrak{t} : [\mu] \to \mathbf{n}$. We identify μ-*tableaux* with labellings of the nodes of $[\mu]$: for example, if $\mu = (1, 3)$ then $\begin{ytableau} 1 \\ 2 & 3 & 4 \end{ytableau}$ is a μ-tableau. If $\mathfrak{t}$ is a μ-tableau we write $Shape(\mathfrak{t}) := \mu$.

We say that a μ-tableau $\mathfrak{t}$ is *row standard* if the entries in $\mathfrak{t}$ increase from left to right in each row and we say that $\mathfrak{t}$ is *standard* if $\mathfrak{t}$ is row standard and the entries also increase from top to bottom. The set of standard λ-tableau is denoted $\mathrm{Std}(\lambda)$ and we write $d_\lambda := |\mathrm{Std}(\lambda)|$ for its cardinality. For example, $\begin{array}{|c|c|c|}\hline 2&3&5\\ \hline 1&4\\ \cline{1-2}\end{array}$ is row standard and $\begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2}\end{array}$ is standard. For a composition of μ of n we denote by $\mathfrak{t}^\mu$ the standard tableau in which the integers $1, 2, \ldots, n$ are entered in increasing order from left to right along the rows of $[\mu]$. For example, if $\mu = (2, 4)$ then $\mathfrak{t}^\mu = \begin{array}{|c|c|c|c|}\cline{1-2} 1&2\\ \hline 3&4&5&6\\ \hline\end{array}$.

The symmetric group $\mathfrak{S}_n$ acts on the right on the set of μ-tableaux by permuting the entries inside a given tableau. The *Young subgroup* associated with μ is the row stabilizer of $\mathfrak{t}^\mu$. Let $\mu = (\mu_1, \ldots, \mu_k)$ and $\nu = (\nu_1, \ldots, \nu_l)$ be compositions. We write $\mu \trianglerighteq \nu$ if for all $i \geq 1$ we have

$$\sum_{j=1}^{i} \mu_j \geq \sum_{j=1}^{i} \nu_j$$

where we add zero parts $\mu_i := 0$ and $\nu_i := 0$ at the end of μ and ν so that the sums are always defined. This is the dominance order on compositions. We extend it to row standard tableaux as follows. Given a row standard tableau $\mathfrak{t}$ of some shape and an integer $m \leq n$, we let $\mathfrak{t} \downarrow m$ be the tableau obtained from $\mathfrak{t}$ by deleting all nodes with entries greater than m. Then, for a pair of μ-tableaux $\mathfrak{s}$ and $\mathfrak{t}$ we write $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if $Shape(\mathfrak{s} \downarrow m) \trianglerighteq Shape(\mathfrak{t} \downarrow m)$ for all $m = 1, \ldots, n$. We write $\mathfrak{s} \vartriangleright \mathfrak{t}$ if $\mathfrak{s} \trianglerighteq \mathfrak{t}$ and $\mathfrak{s} \neq \mathfrak{t}$. This defines the dominance order on tableaux. It is only a partial order, for example

$$\begin{array}{|c|c|}\hline 1&3\\ \hline 2&5\\ \hline 4\\ \cline{1-1}\end{array} \vartriangleright \begin{array}{|c|c|}\hline 2&4\\ \hline 3&5\\ \hline 1\\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1&3\\ \hline 2&5\\ \hline 4\\ \cline{1-1}\end{array} \vartriangleright \begin{array}{|c|c|}\hline 4&5\\ \hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}$$

whereas $\begin{array}{|c|c|}\hline 2&4\\ \hline 3&5\\ \hline 1\\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 4&5\\ \hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}$ are incomparable.

We have that $\mathfrak{t}^\lambda \trianglerighteq \mathfrak{t}$ for all row standard λ-tableau $\mathfrak{t}$.

An r*-multicomposition*, or simply a *multicomposition*, of n is an ordered r-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ of compositions $\lambda^{(k)}$ such that $\sum_{i=1}^{r} |\lambda^{(i)}| = n$. We call $\lambda^{(k)}$ the k'th component of $\boldsymbol{\lambda}$, note that it may be empty. An r*-multipartition*, or simply a *multipartition*, is a multicomposition whose components are partitions. The nodes of a multicomposition are labelled by tuples (x, y, k) with k giving the number of the component and (x, y) the node of that component. For the multicomposition $\boldsymbol{\lambda}$ the set of nodes is denoted $[\boldsymbol{\lambda}]$. This is the Young diagram for $\boldsymbol{\lambda}$ and is represented graphically as the r-tuple of Young diagrams of the components. For example, the Young diagram of $\boldsymbol{\lambda} = ((2,3),(3,1),(1,1,1))$ is

$$\left(\begin{array}{|c|c|c|}\cline{1-2} &\\ \hline &&\\ \hline\end{array}\, , \begin{array}{|c|c|c|}\hline &&\\ \hline \\ \cline{1-1}\end{array}\, , \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array} \right).$$

We denote by $Comp_{r,n}$ the set of r-multicompositions of n and by $Par_{r,n}$ the set of r-multipartitions of n. Let $\boldsymbol{\lambda}$ be a multicomposition of n. A $\boldsymbol{\lambda}$*-multitableau* is a bijection $\mathfrak{t} : [\boldsymbol{\lambda}] \to \mathbf{n}$ which may once again be identified with a filling of $[\boldsymbol{\lambda}]$ using the numbers from $\mathbf{n}$. The restriction of $\mathfrak{t}$ to $\lambda^{(i)}$ is called the i'th component of $\mathfrak{t}$. We say that $\mathfrak{t}$ is *row standard* if all its components are row standard, and *standard* if all its components are standard tableaux. If $\mathfrak{t}$ is a $\boldsymbol{\lambda}$-multitableau we write $Shape(\mathfrak{t}) = \boldsymbol{\lambda}$. The set of all

standard $\boldsymbol{\lambda}$-multitableaux is denoted by $\mathrm{Std}(\boldsymbol{\lambda})$. In the examples

$$\mathfrak{t} = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|}\hline 6 \\ \cline{1-1} 7 \\ \hline 8 & 9 \\ \hline \end{array} \right) \qquad \mathfrak{s} = \left(\begin{array}{|c|c|c|}\hline 2 & 7 & 8 \\ \hline 1 & 4 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|}\hline 5 & 6 \\ \hline \end{array}, \begin{array}{|c|}\hline 3 \\ \hline 9 \\ \hline \end{array} \right) \tag{31}$$

$\mathfrak{t}$ is a standard multitableau whereas $\mathfrak{s}$ is only a row standard tableau. We denote by $\mathfrak{t}^{\boldsymbol{\lambda}}$ the $\boldsymbol{\lambda}$-multitableau in which $1, 2, \ldots, n$ appear in order along the rows of the first component, then along the rows of the second component, and so on. For example, in (31) we have that $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} = ((3,2),(1,1,2))$. For each multicomposition $\boldsymbol{\lambda}$ we define the Young subgroup $\mathfrak{S}_{\boldsymbol{\lambda}}$ as the row stabilizer of $\mathfrak{t}^{\boldsymbol{\lambda}}$.

Let $\mathfrak{s}$ be a row standard $\boldsymbol{\lambda}$-multitableau. We denote by $d(\mathfrak{s})$ the unique element of $\mathfrak{S}_n$ such that $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}} d(\mathfrak{s})$. The set formed by these elements is a complete set of right coset representatives of $\mathfrak{S}_{\boldsymbol{\lambda}}$ in $\mathfrak{S}_n$. Moreover

$$\{d(\mathfrak{s}) \mid \mathfrak{s} \text{ is a row standard } \boldsymbol{\lambda}\text{-multitableau}\}$$

is a *distinguished set of right coset representatives*, that is $\ell(wd(\mathfrak{s})) = \ell(w) + \ell(d(\mathfrak{s}))$ for $w \in \mathfrak{S}_{\boldsymbol{\lambda}}$.

Let $\boldsymbol{\lambda}$ be a multicomposition of n and let $\mathfrak{t}$ be a $\boldsymbol{\lambda}$-multitableau. For $j = 1, \ldots, n$ we write $p_{\mathfrak{t}}(j) := k$ if j appears in the k'th component $\mathfrak{t}^{(k)}$ of $\mathfrak{t}$. We call $p_{\mathfrak{t}}(j)$ the *position* of j in $\mathfrak{t}$. When $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$, we write $p_{\boldsymbol{\lambda}}(j)$ for $p_{\mathfrak{t}^{\boldsymbol{\lambda}}}(j)$ and say that a $\boldsymbol{\lambda}$-multitableau $\mathfrak{t}$ is of the *initial kind* if $p_{\mathfrak{t}}(j) = p_{\boldsymbol{\lambda}}(j)$ for all $j = 1, \ldots, n$.

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r)})$ be multicompositions of n. We write $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ if $\lambda^{(i)} \trianglerighteq \mu^{(i)}$ for all $i = 1, \ldots, n$, this is our dominance order on $Comp_{r,n}$. If $\mathfrak{s}$ and $\mathfrak{t}$ are row standard $\boldsymbol{\lambda}$-multitableaux and $m = 1, \ldots, n$ we define $\mathfrak{s} \downarrow m$ and $\mathfrak{t} \downarrow m$ as for usual tableaux and write $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if $Shape(\mathfrak{s} \downarrow m) \trianglerighteq Shape(\mathfrak{t} \downarrow m)$ for all m.

It should be noted that our dominance order $\trianglerighteq$ is different from the dominance order on multicompositions and multitableaux that is used in some parts of the literature, for example in [11]. Let us denote by $\succeq$ the order used in [11]. Then we have that

$$\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|}\hline 5 \\ \hline \end{array} \right) \succeq \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array} \right)$$

whereas these multitableaux are incomparable with respect to $\trianglerighteq$. On the other hand, if $\mathfrak{s}$ and $\mathfrak{t}$ are multitableaux of the same shape and $p_{\mathfrak{s}}(j) = p_{\mathfrak{t}}(j)$ for all j, then we have that $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if and only if $\mathfrak{s} \succeq \mathfrak{t}$.

To each r-multicomposition $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ we associate a composition $\|\boldsymbol{\lambda}\|$ of length as follows

$$\|\boldsymbol{\lambda}\| := (|\lambda^{(1)}|, \ldots, |\lambda^{(r)}|). \tag{32}$$

3. Tensorial representation of $\mathcal{Y}_{r,n}(q)$

In this section we obtain our first results by constructing a tensor space module for the Yokonuma-Hecke algebra which we show is faithful. From this we deduce that the Yokonuma-Hecke algebra is in fact isomorphic to a specialization of the 'modified Ariki-Koike' algebra, that was introduced by Shoji in [39] and studied for example in [38].

Definition 5 Let V be the free R-module with basis $\{v_i^t \mid 1 \le i \le n,\ 0 \le t \le r-1\}$. Then we define operators $\mathbf{T} \in \mathrm{End}_R(V)$ and $\mathbf{G} \in \mathrm{End}_R(V^{\otimes 2})$ as follows:

$$(v_i^t)\mathbf{T} := \xi^t v_i^t \tag{33}$$

and

$$(v_i^t \otimes v_j^s)\mathbf{G} := \begin{cases} v_j^s \otimes v_i^t & \text{if } t \neq s \\ q v_i^t \otimes v_j^s & \text{if } t = s,\ i = j \\ v_j^s \otimes v_i^t & \text{if } t = s,\ i > j \\ (q - q^{-1}) v_i^t \otimes v_j^s + v_j^s \otimes v_i^t & \text{if } t = s,\ i < j. \end{cases} \tag{34}$$

We extend them to operators $\mathbf{T}_i$ and $\mathbf{G}_i$ acting in the tensor space $V^{\otimes n}$ by letting $\mathbf{T}$ act in the i'th factor and $\mathbf{G}$ in the i'th and $i+1$'st factors, respectively.

Our goal is to prove that these operators define a faithful representation of the Yokonuma-Hecke algebra. We first prove an auxiliary Lemma.

Lemma 6 Let $\mathbf{E}_i$ be defined by $\mathbf{E}_i := \dfrac{1}{r}\sum_{m=0}^{r-1} \mathbf{T}_i^m \mathbf{T}_{i+1}^{-m}$. Consider the map

$$(v_i^t \otimes v_j^s)\mathbf{E} := \begin{cases} 0 & \text{if } t \neq s \\ v_i^t \otimes v_j^s & \text{if } t = s. \end{cases}$$

Then $\mathbf{E}_i$ acts in $V^{\otimes n}$ as $\mathbf{E}$ in the factors $(i, i+1)$ and as the identity in the rest.

PROOF. We have that

$$(v_j^t \otimes v_k^t)\mathbf{T}_i \mathbf{T}_{i+1}^{-1} = \xi^t \xi^{-t} v_j^t \otimes v_k^t = v_j^t \otimes v_k^t.$$

Thus we get immediately that $(v_i^t \otimes v_j^s)\mathbf{E}_i = v_i^t \otimes v_j^s$ if $s = t$. Now, if $s \neq t$ we have that

$$(v_j^t \otimes v_k^s)\mathbf{T}_i \mathbf{T}_{i+1}^{-1} = \xi^t \xi^{-s} v_j^t \otimes v_k^t = \xi^{t-s} v_j^t \otimes v_k^t.$$

Since $0 \leq t, s \leq r-1$, we have that $\xi^{t-s} \neq 1$ which implies that

$$\sum_{m=0}^{r-1} \xi^{m(t-s)} = (\xi^{r(t-s)} - 1)/(\xi^{(t-s)} - 1) = 0$$

and so it follows that $(v_i^t \otimes v_j^s)\mathbf{E} = 0$ if $s \neq t$. $\square$

Remark 7 The operators $\mathbf{G}_i$ and $\mathbf{E}_i$ should be compared with the operators introduced in [36] in order to obtain a representation of $\mathcal{E}_n(q)$ in $V^{\otimes n}$. Let us denote by $\widetilde{\mathbf{G}}_i$ and $\widetilde{\mathbf{E}}_i$ the operators defined in [36]. Then we have that $\mathbf{E}_i = \widetilde{\mathbf{E}}_i$ and

$$\mathbf{G} = \widetilde{\mathbf{G}}_i + (q^{-1} - 1)\mathbf{E}_i \widetilde{\mathbf{G}}_i$$

corresponding to the change of presentation given in (30).

Theorem 8 There is a representation ρ of $\mathcal{Y}_{r,n}(q)$ in $V^{\otimes n}$ given by $t_i \to \mathbf{T}_i$ and $g_i \to \mathbf{G}_i$.

PROOF. We must check that the operators $\mathbf{T}_i$ and $\mathbf{G}_i$ satisfy the relations (4),…,(9) of the Yokonuma-Hecke algebra. Here the relations (4) and (5) are trivially satisfied since the $\mathbf{T}_i$'s commute. The relation (7) is also easy to verify since the operators $\mathbf{G}_i$ and $\mathbf{G}_j$ act as $\mathbf{G}$ in two different consecutive factors if $|i - j| > 1$.

In order to prove the braid relations (8) we rely on the fact, obtained in [36] Theorem 1, that the operators $\widetilde{\mathbf{G}}_i$'s and $\mathbf{E}_i$'s satisfy the relations for the algebra of braids and ties $\mathcal{E}_n(q)$. Indeed, via Remark 7 we get from this that

$$\begin{aligned} \mathbf{G}_i \mathbf{G}_{i+1} \mathbf{G}_i &= \\ &= (1 + (q^{-1} - 1)\mathbf{E}_i)(1 + (q^{-1} - 1)\mathbf{E}_{i,i+2})(1 + (q^{-1} - 1)\mathbf{E}_{i+1})\widetilde{\mathbf{G}}_i \widetilde{\mathbf{G}}_{i+1} \widetilde{\mathbf{G}}_i \\ &= (1 + (q^{-1} - 1)\mathbf{E}_i)(1 + (q^{-1} - 1)\mathbf{E}_{i,i+2})(1 + (q^{-1} - 1)\mathbf{E}_{i+1})\widetilde{\mathbf{G}}_{i+1} \widetilde{\mathbf{G}}_i \widetilde{\mathbf{G}}_{i+1} \\ &= \mathbf{G}_{i+1} \mathbf{G}_i \mathbf{G}_{i+1} \end{aligned}$$

and (8) follows as claimed. In a similar way we get that the $\mathbf{G}_i$'s satisfy the quadratic relation (9).

We are then only left with the relation (6). We have here three cases to consider:

$$\mathbf{T}_i\mathbf{G}_j = \mathbf{G}_j\mathbf{T}_i \qquad |i-j|>1 \tag{35}$$

$$\mathbf{T}_i\mathbf{G}_i = \mathbf{G}_i\mathbf{T}_{i+1} \tag{36}$$

$$\mathbf{T}_{i+1}\mathbf{G}_i = \mathbf{G}_i\mathbf{T}_i. \tag{37}$$

The case (35) clearly holds since the operators $\mathbf{T}_i$ and $\mathbf{G}_j$ act in different factors of the tensor product $v_{i_1}^{j_1}\otimes v_{i_2}^{j_2}\otimes\ldots\otimes v_{i_n}^{j_n}$. In order to verify the other two cases we may assume that $i=1$ and $n=2$. It is enough to evaluate on vectors of the form $v_{i_1}^{j_1}\otimes v_{i_2}^{j_2}\in V^{\otimes 2}$. For $j_1=j_2$ the actions of $\mathbf{T}_1$ and $\mathbf{T}_2$ are given as the multiplication with the same scalar and so the relations (36) and (37) also hold.

Suppose then finally that $j_1\neq j_2$. We then have that

$$(v_{i_1}^{j_1}\otimes v_{i_2}^{j_2})\mathbf{T}_1\mathbf{G}_1 = \xi^{j_1} v_{i_2}^{j_2}\otimes v_{i_1}^{j_1} = (v_{i_1}^{j_1}\otimes v_{i_2}^{j_2})\mathbf{G}_1\mathbf{T}_2$$

and

$$(v_{i_1}^{j_1}\otimes v_{i_2}^{j_2})\mathbf{T}_2\mathbf{G}_1 = \xi^{j_2} v_{i_2}^{j_2}\otimes v_{i_1}^{j_1} = (v_{i_1}^{j_1}\otimes v_{i_2}^{j_2})\mathbf{G}_1\mathbf{T}_1$$

and the proof of the Theorem is finished. □

Remark 9 Let $\mathcal{K}$ be an R-algebra as in the previous section with corresponding specialized Yokonuma-Hecke algebra $\mathcal{Y}_{r,n}^{\mathcal{K}}(q)$. Then we obtain a specialized tensor product representation $\rho^{\mathcal{K}}:\mathcal{Y}_{r,n}^{\mathcal{K}}(q)\to \mathrm{End}_{\mathcal{K}}(V^{\otimes n})$. Indeed, the above proof amounts only to checking relations, and so carries over to $\mathcal{Y}_{r,n}^{\mathcal{K}}(q)$.

Theorem 10 ρ and $\rho^{\mathcal{K}}$ are faithful representations.

Proof. We first consider the faithfulness of ρ. Recall Juyumaya's R-basis for $\mathcal{Y}_{r,n}(q)$

$$\mathcal{B}_{r,n} = \{g_\sigma t_1^{j_1}\cdots t_n^{j_n}\mid \sigma\in\mathfrak{S}_n,\ j_i\in\mathbb{Z}/r\mathbb{Z}\}.$$

For $\sigma = s_{i_1}\ldots s_{i_m}\in\mathfrak{S}_n$ written in reduced form we define $\mathbf{G}_\sigma := \mathbf{G}_{i_1}\ldots\mathbf{G}_{i_m}$. To prove that ρ is faithful it is enough to show that

$$\rho(\mathcal{B}_{r,n}) = \{\mathbf{G}_\sigma\mathbf{T}_1^{j_1}\cdots\mathbf{T}_n^{j_n}\mid \sigma\in\mathfrak{S}_n,\ j_k\in\mathbb{Z}/r\mathbb{Z}\}$$

is an R-linearly independent subset of $\mathrm{End}(V^{\otimes n})$. Suppose therefore that there exists a nontrivial linear dependence

$$\sum_{\substack{\sigma\in\mathfrak{S}_n\\ j_i\in\mathbb{Z}/r\mathbb{Z}}} \lambda_{j_1,\ldots,j_n,\sigma}\mathbf{G}_\sigma\mathbf{T}_1^{j_1}\cdots\mathbf{T}_n^{j_n}=0 \tag{38}$$

where not every $\lambda_{j_1,\ldots,j_n,\sigma}\in R$ is zero.

We first observe that for arbitrary a_i's and $\sigma\in\mathfrak{S}_n$ the action of $\mathbf{G}_\sigma$ on the special tensor $v_n^{a_n}\otimes\cdots\otimes v_1^{a_1}$, having the lower indices strictly decreasing, is particularly simple. Indeed, since $\sigma = s_{i_1}\ldots s_{i_m}$ is a reduced expression for σ we have that the action of $\mathbf{G}_\sigma = \mathbf{G}_{i_1}\ldots\mathbf{G}_{i_m}$ in that case always involves the third case of (34) and thus is given by place permutation, in other words

$$(v_n^{a_n}\otimes\cdots\otimes v_1^{a_1})\mathbf{G}_\sigma = (v_n^{a_n}\otimes\cdots\otimes v_1^{a_1})\sigma = v_{i_n}^{a_{i_n}}\otimes\cdots\otimes v_{i_1}^{a_{i_1}} \tag{39}$$

for some permutation $i_n, \dots, i_1$ of $n, \dots, 1$ uniquely given by σ. Let $\mathfrak{T}_n$ be the R-subalgebra of $\mathrm{End}(V^{\otimes n})$ generated by the $\mathbf{T}_i$'s. For fixed $k_1, \dots, k_n$ we now define

$$V_{k_1,\dots,k_n} := \mathrm{Span}_R\{v_{k_1}^{j_1} \otimes \cdots \otimes v_{k_n}^{j_n} \mid j_k \in \mathbb{Z}/r\mathbb{Z}\}.$$

Then $V_{k_1,\dots,k_n}$ is a $\mathfrak{T}_n$-submodule of $V^{\otimes n}$. Given (39), to prove that the linear dependence (38) does not exist, it is now enough to show that $V_{k_1,\dots,k_n}$ is a faithful $\mathfrak{T}_n$-module.

For $j = 0, 1, \dots, r-1$ we define $w_k^j \in V$ via

$$w_k^j := \sum_{i=0}^{r-1} \xi^{ij} v_k^i.$$

Then $\{w_k^i \mid i = 0, 1, \dots, r-1, k = 1, \dots, n\}$ is also an R-basis for V, since for fixed k the base change matrix between $\{v_k^i \mid i = 0, 1, \dots, r-1\}$ and $\{w_k^j \mid j = 0, 1, \dots, r-1\}$ is given by a Vandermonde matrix with determinant $\prod_{0 \le i < j \le r-1} (\xi^i - \xi^j)$ which is a unit in R. But then also $\{w_{k_1}^{j_1} \otimes \ldots \otimes w_{k_n}^{j_n} | j_i \in \mathbb{Z}/r\mathbb{Z}\}$ is an R-basis for $V_{k_1,\dots,k_n}$. On the other hand, for all j we have that $\mathbf{T} w_k^j = w_k^{j+1}$ where the indices are understood modulo r. Hence, given the nontrivial linear combination in $\mathfrak{T}_n$

$$\sum_{j_i \in \mathbb{Z}/r\mathbb{Z}} \lambda_{j_1,\dots,j_n} \mathbf{T}_1^{j_1} \cdots \mathbf{T}_n^{j_n}$$

we get by acting with it on $w_{k_1}^0 \otimes \ldots \otimes w_{k_n}^0$ the following nonzero element

$$\sum_{j_i \in \mathbb{Z}/r\mathbb{Z}} \lambda_{j_1,\dots,j_n} w_{k_1}^{j_1} \otimes \ldots \otimes w_{k_n}^{j_n}.$$

This proves the Theorem in the case of ρ. The case $\rho^{\mathcal{K}}$ is proved similarly, using that $\prod_{0 \le i < j \le r-1} (\xi^i - \xi^j)$ is a unit in $\mathcal{K}$ as well.

$\square$

3.1. The modified Ariki-Koike algebra.

In this subsection we obtain our first main result, showing that the Yokonuma-Hecke algebra is isomorphic to a variation of the Ariki-Koike algebra, called the modified Ariki-Koike algebra $\mathcal{H}_{r,n}$. It was introduced by Shoji. Given the faithful tensor representation of the previous subsection, the proof of this isomorphism Theorem is actually almost trivial, but still we think that it is a surprising result. Indeed, the quadratic relations involving the braid group generators look quite different in the two algebras and as a matter of fact the usual Hecke algebra of type A_{n-1} appears naturally as a subalgebra of the (modified) Ariki-Koike algebra, but only as quotient of the Yokonuma-Hecke algebra.

Let us recall Shoji's definition of the modified Ariki-Koike algebra. He defined it over the ring $R_1 := \mathbb{Z}[q, q^{-1}, u_1, \dots, u_r, \Delta^{-1}]$, where $q, u_1, \dots, u_r$ are indeterminates and $\Delta := \prod_{i>j} (u_i - u_j)$ is the Vandermonde determinant. We here consider the modified Ariki-Koike algebra over the ring R, corresponding to a specialization of Shoji's algebra via the homomorphism $\varphi : R_1 \to R$ given by $u_i \mapsto \xi^i$ and $q \mapsto q$.

Let $\mathbf{A}$ be the square matrix of degree r whose ij-entry is given by $\mathbf{A}_{ij} = \xi^{j(i-1)}$ for $1 \le i, j \le r$, i.e. $\mathbf{A}$ is the usual Vandermonde matrix. Then we can write the inverse of

$\mathbf{A}$ as $\mathbf{A}^{-1} = \Delta^{-1}\mathbf{B}$, where $\Delta = \prod_{i>j}(\xi^i - \xi^j)$ and $\mathbf{B} = (h_{ij})$ is the adjoint matrix of $\mathbf{A}$, and for $1 \le i \le r$ define a polynomial $F_i(X) \in \mathbb{Z}[\xi][X] \subseteq R[X]$ by

$$F_i(X) := \sum_{1 \le i \le r} h_{ij} X^{j-1}.$$

Definition 11 The modified Ariki-Koike algebra, denoted $\mathcal{H}_{r,n} = \mathcal{H}_{r,n}(q)$, is the associative R-algebra generated by the elements $h_2, \ldots, h_n$ and $\omega_1, \ldots, \omega_n$ subject to the following relations:

$$(h_i - q)(h_i + q^{-1}) = 0 \qquad \text{for all } i \tag{40}$$
$$h_i h_j = h_j h_i \qquad \text{for } |i-j| > 1 \tag{41}$$
$$h_i h_{i+1} h_i = h_{i+1} h_i h_{i+1} \qquad \text{for all } i = 1, \ldots, n-2 \tag{42}$$
$$(\omega_i - \xi^1) \cdots (\omega_i - \xi^r) = 0 \qquad \text{for all } i \tag{43}$$
$$\omega_i \omega_j = \omega_j \omega_i \qquad \text{for all } i, j \tag{44}$$

$$h_j \omega_j = \omega_{j-1} h_j + \Delta^{-2} \sum_{c_1 < c_2} (\xi^{c_2} - \xi^{c_1})(q - q^{-1}) F_{c_1}(\omega_{j-1}) F_{c_2}(\omega_j) \tag{45}$$
$$h_j \omega_{j-1} = \omega_j h_j - \Delta^{-2} \sum_{c_1 < c_2} (\xi^{c_2} - \xi^{c_1})(q - q^{-1}) F_{c_1}(\omega_{j-1}) F_{c_2}(\omega_j) \tag{46}$$
$$h_j \omega_l = \omega_l h_j \qquad l \neq j, j-1 \tag{47}$$

$\mathcal{H}_{r,n}(q)$ was introduced as a way of approximating the usual Ariki-Koike algebra and is isomorphic to it if a certain separation condition holds. In general the two algebras are not isomorphic, but related via a, somewhat mysterious, homomorphism from the Ariki-Koike algebra to $\mathcal{H}_{r,n}(q)$, see [39].

Sakamoto and Shoji, [39] and [38], gave a $\mathcal{H}_{r,n}(q)$-module structure on $V^{\otimes n}$ that we now explain. We first introduce a total order on the v_i^j's via

$$v_1^1, v_2^1, \ldots, v_n^1, v_1^2, \ldots, v_n^2, \ldots, v_1^r, \ldots, v_n^r \tag{48}$$

and denote by $v_1, \ldots, v_{rn}$ these vectors in this order. We then define the linear operator $\mathbf{H} \in \mathrm{End}(V^{\otimes 2})$ as follows:

$$(v_i \otimes v_j)\mathbf{H} := \begin{cases} q v_i \otimes v_j & \text{if } i = j \\ v_j \otimes v_i & \text{if } i > j \\ (q - q^{-1}) v_i \otimes v_j + v_j \otimes v_i & \text{if } i < j. \end{cases}$$

We then extend this to an operator $\mathbf{H}_i$ of $V^{\otimes n}$ by letting $\mathbf{H}$ act in the i'th and $i+1$'st factors. This is essentially Jimbo's original operator for constructing tensor representations for the usual Iwahori-Hecke algebra $\mathcal{H}_n$ of type A. The following result is shown in [39].

Theorem 12 The map $\tilde{\rho} : \mathcal{H}_{r,n}(q) \to \mathrm{End}(V^{\otimes n})$ given by $h_j \to \mathbf{H}_i$, $\omega_j \to \mathbf{T}_j$ defines a faithful representation of $\mathcal{H}_{r,n}(q)$.

We are now in position to prove the following main Theorem.

Theorem 13 The Yokonuma-Hecke algebra $\mathcal{Y}_{r,n}(q)$ is isomorphic to the modified Ariki-Koike algebra $\mathcal{H}_{r,n}(q)$.

PROOF. By the previous Theorem and Theorem 10 we can identify $\mathcal{Y}_{r,n}(q)$ and $\mathcal{H}_{r,n}(q)$ with the subalgebras $\rho(\mathcal{Y}_{r,n}(q))$ and $\tilde{\rho}(\mathcal{H}_{r,n}(q))$ of $\mathrm{End}(V^{\otimes n})$, respectively. Hence, in order to prove the Theorem we must show that $\rho(\mathcal{Y}_{r,n}(q)) = \tilde{\rho}(\mathcal{H}_{r,n}(q))$. But by definition, we surely have that the $\mathbf{T}_i$'s belong to both subalgebras, since $\mathbf{T}_i = \rho(t_i)$ and $\mathbf{T}_i = \tilde{\rho}(\omega_i)$.

It is therefore enough to show that the $\mathbf{G}_i$'s from $\rho(\mathcal{Y}_{r,n}(q))$ belong to $\tilde{\rho}(\mathcal{H}_{r,n})$, and that the $\mathbf{H}_i$'s from $\tilde{\rho}(\mathcal{H}_{r,n})$ belong to $\rho(\mathcal{Y}_{r,n}(q))$.

On the other hand, the operator $\mathbf{G}$ coincides with the operator denoted by S in Shoji's paper [39]. But then Lemma 3.5 of that paper is the equality

$$\mathbf{G}_{i-1} = \mathbf{H}_i - \Delta^{-2}(q-q^{-1}) \sum_{c_1<c_2} F_{c_1}(\mathbf{T}_{i-1}) F_{c_2}(\mathbf{T}_i).$$

Thus, since $\Delta^{-2}(q-q^{-1})\sum_{c_1<c_2} F_{c_1}(\mathbf{T}_{i-1})F_{c_2}(\mathbf{T}_i)$ belongs to both algebras $\tilde{\rho}(\mathcal{H}_{r,n}(q))$ and $\rho(\mathcal{Y}_{r,n}(q))$, the Theorem follows. $\square$

Lusztig gave in [27] a structure Theorem for $\mathcal{Y}_{r,n}(q)$, showing that it is a direct sum of matrix algebras over Iwahori-Hecke algebras of type A. This result was recently recovered by Jacon and Poulain d'Andecy in [21]. We now briefly explain how this result, via our isomorphism Theorem, is equivalent to a similar result for $\mathcal{H}_{r,n}(q)$, obtained in [20] and [39]

For a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ of n of length r, we let $\mathcal{H}_\mu(q)$ be the corresponding *Young-Hecke algebra*, by which we mean that $\mathcal{H}_\mu(q)$ is the R-subalgebra of $\mathcal{H}_n(q)$ generated by the g_i's for $i \in \Sigma_n \cap \mathfrak{S}_\mu$. Thus $\mathcal{H}_\mu(q) = \mathcal{H}_{\mu_1}(q) \otimes \ldots \otimes \mathcal{H}_{\mu_r}(q)$ where each factor $\mathcal{H}_{\mu_i}(q)$ is a Iwahori-Hecke algebra corresponding to the indices given by the part μ_i. Let p_μ denote the multinomial coefficient

$$p_\mu := \binom{n}{\mu_1 \cdots \mu_r}.$$

With this notation, the structure Theorem due to Lusztig and Jacon-Poulain d'Andecy is as follows

$$\mathcal{Y}_{r,n}(q) \cong \bigoplus_{\mu=(\mu_1,\mu_2,\ldots,\mu_r)\models n} \mathrm{Mat}_{p_\mu}(\mathcal{H}_\mu(q)) \tag{49}$$

where for any R-algebra $\mathcal{A}$, we denote by $\mathrm{Mat}_m(\mathcal{A})$ the $m \times m$ matrix algebra with entries in $\mathcal{A}$.

On the other hand, a similar structure Theorem was established for the modified Ariki-Koike algebra $\mathcal{H}_{r,n}(q)$, independently by Sawada and Shoji in [38] and by Hu and Stoll in [20]:

$$\mathcal{H}_{r,n}(q) \cong \bigoplus_{\mu=(\mu_1,\mu_2,\ldots,\mu_r)\models n} \mathrm{Mat}_{p_\mu}(\mathcal{H}_\mu(q)). \tag{50}$$

Thus, our isomorphism Theorem 13 shows that above two structure Theorems are equivalent.

We finish this section by showing the following embedding Theorem, already announced above. It is also a consequence of our tensor space module for $\mathcal{Y}_{r,n}(q)$.

Theorem 14 Suppose that $r \geq n$. Then the homomorphism $\varphi : \mathcal{E}_n^{\mathcal{K}}(q) \to \mathcal{Y}_{r,n}^{\mathcal{K}}(q)$ introduced in Lemma 4 is an embedding.

Remark 15 In the case $\mathcal{K} = \mathbb{C}[q, q^{-1}]$ and $r = n$ the Theorem is an immediate consequence of the faithfulness of the $\mathcal{E}_n(q)$-module $V^{\otimes n}$ proved in Corollary 4 of [36] since the injectivity of the tensor space representation $\rho^{\mathcal{K}}_{\mathcal{E}_n} : \mathcal{E}_n(q) \to \mathrm{End}(V^{\otimes n})$ together with the factorization $\rho^{\mathcal{K}}_{\mathcal{E}_n} = \rho^{\mathcal{K}} \circ \varphi$ implies that φ is injective. Actually one easily checks that the proof of Corollary 4 of [36] is also valid for $\mathcal{K} = R$ and $r \geq n$, but still this does not give the injectivity of φ for general $\mathcal{K}$ since extension of scalars from R to $\mathcal{K}$ is not left exact. Note that the specialization argument of [36] would fail for general $\mathcal{K}$.

In order to prove Theorem 14 we need to modify the proof of Corollary 4 of [36] to make it valid for general $\mathcal{K}$. For this we first prove the following Lemma.

Lemma 16 Suppose that $r \geq n$. Let $\mathcal{K}$ be an R-algebra as above and let $A = (I_1, \dots, I_d) \in \mathcal{SP}_n$ be a set partition. Denote by V_A the $\mathcal{K}$-submodule of $V^{\otimes n}$ spanned by the vectors

$$v_n^{j_n} \otimes \cdots \otimes v_k^{j_k} \otimes \cdots \otimes v_l^{j_l} \otimes \cdots \otimes v_1^{j_1} \qquad 0 \leq j_k \leq r-1$$

with decreasing lower indices and satisfying that $j_k = j_l$ exactly when if k and l belong to the same block I_i of A. Let $E_A \in \mathcal{E}_n^{\mathcal{K}}(q)$ be the element defined the same way as $E_A \in \mathcal{Y}_{r,n}(q)$, that is via formula (20). Then for all $v \in V_A$ we have that $vE_A = v$ whereas $vE_B = 0$ for $B \in \mathcal{SP}_n$ satisfying $B \not\subseteq A$ with respect to the order $\subseteq$ introduced above.

Proof. Note first that the condition $r \geq n$ ensures that $V_A \neq 0$. In order to prove the first statement it is enough to show that e_{kl} acts as the identity on the basis vectors of V_A whenever k and l belong to the same block of A. But this follows from the expression for e_{kl} given in (15) together with the definition (34) of the action of $\mathbf{G}_i$ on $V^{\otimes n}$ and Lemma 6. Just as in the proof of Theorem 10 we use that the action of $\mathbf{G}_i$ on $v \in V_A$ is just permutation of the i'th and $i+1$'st factors of v since the lower indices are decreasing.

In order to show the second statement, we first remark that the condition $B \not\subseteq A$ means that there exist i and j belonging to the same block of B, but to different blocks of A. In other words e_{ij} appears as a factor of the product defining E_B whereas for all basis vectors of V_A

$$v_n^{j_n} \otimes \cdots \otimes v_k^{j_k} \otimes \cdots \otimes v_l^{j_l} \otimes \cdots \otimes v_1^{j_1}$$

we have that $j_k \neq j_l$. Just as above, using that the action of $\mathbf{G}_i$ is given by place permutation when the lower indices are decreasing, we deduce from this that $V_A e_{ij} = 0$ and so finally that $V_A e_B = 0$, as claimed. □

Proof of Theorem 14. Recall from Theorem 2 of [36] that the set $\{E_A g_w | A \in \mathcal{SP}_n, w \in \mathfrak{S}_n\}$ generates $\mathcal{E}_n(q)$ over $\mathbb{C}[q, q^{-1}]$ (it is even a basis). The proof of this does not involve any special properties of $\mathbb{C}$ and hence $\{E_A g_w | A \in \mathcal{SP}_n, w \in \mathfrak{S}_n\}$ also generates $\mathcal{E}_n^{\mathcal{K}}(q)$ over $\mathcal{K}$.

Let us now consider a nonzero element $\sum_{w,A} r_{w,A} E_A G_w$ in $\mathcal{E}_n^{\mathcal{K}}(q)$. It is mapped under $\varphi_{\mathcal{E}_n^{\mathcal{K}}}$ to $\sum_{w,A} r_{w,A} \mathbf{E}_A \mathbf{G}_w$ which we must show to be nonzero.

For this we choose $A_0 \in \mathcal{SP}_n$ satisfying $r_{w,A_0} \neq 0$ for some $w \in \mathfrak{S}_n$ and minimal with respect to this under our order $\subseteq$ on $\mathcal{SP}_n$. Let $v \in V_{A_0}$ where V_{A_0} is defined as in the previous Lemma 16. Then the Lemma gives us that

$$v\Big(\sum_{w,A} r_{w,A} \mathbf{E}_A \mathbf{G}_w\Big) = v\Big(\sum_{w} r_{w,A_0} \mathbf{G}_w\Big). \tag{51}$$

The lower indices of v are strictly decreasing and so each $\mathbf{G}_w$ acts on it by place permutation. It follows from this that (51) is nonzero, and the Theorem is proved. □

Remark 17 The above proof did not use the linear independence of $\{E_A g_w | A \in \mathcal{SP}_n, w \in \mathfrak{S}_n\}$ over $\mathcal{K}$. In fact, it gives a new proof of Corollary 4 of [36].

4. CELLULAR BASIS FOR THE YOKONUMA-HECKE ALGEBRA

The goal of this section is to construct a cellular basis for the Yokonuma-Hecke algebra. The cellularity of the Yokonuma-Hecke algebra could also have been obtained from the cellularity of the modified Ariki-Koike algebra, see [38], via our isomorphism Theorem from the previous section. We have several reasons for still giving a direct construction of a cellular basis for the Yokonuma-Hecke algebra. Firstly, we believe that our construction is simpler and more natural than the one in [38]. Secondly, our basis turns out to have a nice compatiblity property with the subalgebra $\mathfrak{T}_n$ of $\mathcal{Y}_{r,n}(q)$ studied above, a compatibility that we would like to emphasize. This compatibiliy is essential for our proof of Lusztig's presentation for $\mathcal{Y}_{r,n}(q)$, given at the end of this section. We also need the cellular basis in order to show, in the following section, that the Jucys-Murphy operators introduced by Chlouveraki and Poulain d'Andecy are JM-elements in the abstract sense introduced by Mathas. Finally, several of the methods for the construction of the basis are needed in the last section where the algebra of braids and ties is treated.

Let us start out by recalling the definition from [14] of a cellular basis.

Definition 18 Let $\mathcal{R}$ be an integral domain. Suppose that A is an $\mathcal{R}$-algebra which is free as an $\mathcal{R}$-module. Suppose that $(\Lambda, \geq)$ is a poset and that for each $\lambda \in \Lambda$ there is a finite indexing set $T(\lambda)$ (the 'λ-tableaux') and elements $c^{\lambda}_{\mathfrak{s}\mathfrak{t}} \in A$ such that

$$\mathcal{C} = \{c^{\lambda}_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Lambda \text{ and } \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$$

is an $\mathcal{R}$-basis of A. The pair $(\mathcal{C}, \Lambda)$ is a *cellular basis* of A if

- **(i)** The $\mathcal{R}$-linear map $* : A \to A$ determined by $(c^{\lambda}_{\mathfrak{s}\mathfrak{t}})^* = c^{\lambda}_{\mathfrak{t}\mathfrak{s}}$ for all $\lambda \in \lambda$ and all $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ is an algebra anti-automorphism of A.
- **(ii)** For any $\lambda \in \Lambda$, $\mathfrak{t} \in T(\lambda)$ and $a \in A$ there exist $r_{\mathfrak{v}} \in R$ such that for all $\mathfrak{s} \in T(\lambda)$

$$c^{\lambda}_{\mathfrak{s}\mathfrak{t}} a \equiv \sum_{\mathfrak{v} \in T(\lambda)} r_{\mathfrak{v}} c^{\lambda}_{\mathfrak{s}\mathfrak{v}} \mod A^{\lambda}$$

  where A^{λ} is the $\mathcal{R}$-submodule of A with basis $\{c^{\mu}_{\mathfrak{u}\mathfrak{v}} \mid \mu \in \Lambda, \mu > \lambda \text{ and } \mathfrak{u}, \mathfrak{v} \in T(\mu)\}$.

If A has a cellular basis we say that A is a *cellular algebra*.

For our cellular basis for $\mathcal{Y}_{r,n}(q)$ we use for Λ the set $Par_{r,n}$ of r-multipartitions of n, endowed with the dominance order as explained in section 2, and for $T(\lambda)$ we use the set of standard r-multitableaux $\mathrm{Std}(\boldsymbol{\lambda})$, introduced in the same section. For $* : \mathcal{Y}_{r,n}(q) \to \mathcal{Y}_{r,n}(q)$ we use the R-linear antiautomorphism of $\mathcal{Y}_{r,n}(q)$ determined by $g_i^* = g_i$ and $t_k^* = t_k$ for $1 \leq i < n$ and $1 \leq k \leq n$. Note that $*$ does exist as can easily be checked from the relations defining $\mathcal{Y}_{r,n}(q)$.

We then only have to explain the construction of the basis element itself, for pairs of standard tableaux. Our guideline for this is Murphy's construction of *the standard basis* of the Iwahori-Hecke algebra $\mathcal{H}_n(q)$.

For $\boldsymbol{\lambda} \in Comp_{r,n}$ we first define

$$x_{\boldsymbol{\lambda}} := \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} q^{\ell(w)} g_w. \tag{52}$$

In the case of the Iwahori-Hecke algebra $\mathcal{H}_n(q)$, and $\boldsymbol{\lambda}$ a usual composition, the element $x_{\boldsymbol{\lambda}}$ is the starting point of Murphy's standard basis, corresponding to the most

dominant tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$. In our more complicated case $\mathcal{Y}_{r,n}(q)$, the element $x_{\boldsymbol{\lambda}}$ will only be the first ingredient of the $\mathcal{Y}_{r,n}(q)$-element corresponding to the tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$. Let us now explain the other two ingredients.

For a composition $\mu = (\mu_1, \dots, \mu_k)$ we define the *reduced composition* $\operatorname{red}\mu$ as the composition obtained from μ by deleting all zero parts $\mu_i = 0$ from μ. We say that a composition μ is reduced if $\mu = \operatorname{red}\mu$.

For any reduced composition $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ we introduce the set partition $A_\mu := (I_1, I_2, \dots, I_k)$ by filling in the numbers consecutively, that is

$$I_1 := \{1, 2, \dots, \mu_1\},\ I_2 := \{\mu_1 + 1, \mu_1 + 2, \dots, \mu_1 + \mu_2\},\ etc. \tag{53}$$

and for a multicomposition $\boldsymbol{\lambda} \in Comp_{r,n}$ we define $A_{\boldsymbol{\lambda}} := A_{\operatorname{red}\|\boldsymbol{\lambda}\|} \in \mathcal{SP}_n$. Thus we get for any $\boldsymbol{\lambda} \in Comp_{r,n}$ an idempotent $E_{A_{\boldsymbol{\lambda}}} \in \mathcal{Y}_{r,n}(q)$ which will be the second ingredient of our $\mathcal{Y}_{r,n}(q)$-element for $\mathfrak{t}^{\boldsymbol{\lambda}}$. Clearly $t_i E_{A_{\boldsymbol{\lambda}}} = E_{A_{\boldsymbol{\lambda}}} t_i$ for all i. Moreover $E_{A_{\boldsymbol{\lambda}}}$ satisfies the following key property.

Lemma 19 Let $\boldsymbol{\lambda} \in Comp_{r,n}$ and let $A_{\boldsymbol{\lambda}}$ be the associated set partition. Suppose that k and l belong to the same block of $A_{\boldsymbol{\lambda}}$. Then $t_k E_{A_{\boldsymbol{\lambda}}} = t_l E_{A_{\boldsymbol{\lambda}}}$.

PROOF. This follows from the definitions. □

From Juyumaya's basis (13) it follows that t_i is a diagonalizable element on $\mathcal{Y}_{r,n}(q)$. The eigenspace projector for the action t_i on $\mathcal{Y}_{r,n}(q)$ with eigenvalue ξ^k is

$$u_{ik} = \frac{1}{r}\sum_{j=0}^{r-1} \xi^{-jk} t_i^j \in \mathcal{Y}_{r,n}(q) \tag{54}$$

that is $\{v \in \mathcal{Y}_{r,n}(q) | t_i v = \xi^k v\} = u_{ik}\mathcal{Y}_{r,n}(q)$. For $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in Comp_{r,n}$ we define $U_{\boldsymbol{\lambda}}$ as the product

$$U_{\boldsymbol{\lambda}} := \prod_{j=1}^{r} u_{i_j, j} \tag{55}$$

where i_j is any number from the j'th component of $\mathfrak{t}^{\boldsymbol{\lambda}}$. We have now gathered all the ingredients of our $\mathcal{Y}_{r,n}(q)$-element corresponding to $\mathfrak{t}^{\boldsymbol{\lambda}}$.

Definition 20 Let $\boldsymbol{\lambda} \in Comp_{r,n}$. Then we define $m_{\boldsymbol{\lambda}} \in \mathcal{Y}_{r,n}(q)$ via

$$m_{\boldsymbol{\lambda}} := U_{\boldsymbol{\lambda}} E_{A_{\boldsymbol{\lambda}}} x_{\boldsymbol{\lambda}}. \tag{56}$$

The following Lemmas contain some basic properties for $m_{\boldsymbol{\lambda}}$.

Lemma 21 The following properties for $m_{\boldsymbol{\lambda}}$ are true.

(1) The element $m_{\boldsymbol{\lambda}}$ is independent of the choices of i_j's.
(2) For i in the j'th component of $\mathfrak{t}^{\boldsymbol{\lambda}}$ (that is $p_{\boldsymbol{\lambda}}(i) = j$) we have $t_i m_{\boldsymbol{\lambda}} = m_{\boldsymbol{\lambda}} t_i = \xi^j m_{\boldsymbol{\lambda}}$.
(3) The factors $U_{\boldsymbol{\lambda}}$, $E_{A_{\boldsymbol{\lambda}}}$ and $x_{\boldsymbol{\lambda}}$ of $m_{\boldsymbol{\lambda}}$ commute with each other.
(4) If i and j occur in the same block of $A_{\boldsymbol{\lambda}}$ then $m_{\boldsymbol{\lambda}} e_{ij} = e_{ij} m_{\boldsymbol{\lambda}} = m_{\boldsymbol{\lambda}}$.
(5) If i and j occur in two different blocks of $A_{\boldsymbol{\lambda}}$ then $m_{\boldsymbol{\lambda}} e_{ij} = 0 = e_{ij} m_{\boldsymbol{\lambda}}$.
(6) For all $w \in \mathfrak{S}_{\boldsymbol{\lambda}}$ we have $m_{\boldsymbol{\lambda}} g_w = g_w m_{\boldsymbol{\lambda}} = q^{\ell(w)} m_{\boldsymbol{\lambda}}$.

PROOF. The properties (1) and (2) are consequences of the definitions, whereas (3) follows from (2) and Lemma 2. Properties (4) and (5) follow from (2) and (3). Finally, to show (6) we note that for $s_i \in \mathfrak{S}_{\boldsymbol{\lambda}}$ we have that $E_{A_{\boldsymbol{\lambda}}} g_i^2 = E_{A_{\boldsymbol{\lambda}}}(1+(q-q^{-1})g_i)$. Thus the statement of (6) reduces to the similar Iwahori-Hecke algebra statement for $x_{\boldsymbol{\lambda}}$ which can be found for instance in [32, Lemma 3.2]. □

Remark 22 Note that i and j are in the same block of $\mathcal{A}_{\boldsymbol{\lambda}}$ if and only if they are in the same component of $\mathfrak{t}^{\boldsymbol{\lambda}}$. However, the enumerations of the blocks of $\mathcal{A}_{\boldsymbol{\lambda}}$ and the components of $\mathfrak{t}^{\boldsymbol{\lambda}}$ are different since $\mathfrak{t}^{\boldsymbol{\lambda}}$ may have empty components and so in part (2) of the Lemma we cannot replace one by the other.

Lemma 23 Let $\boldsymbol{\lambda} \in Comp_{r,n}$ and suppose that $w \in \mathfrak{S}_n$. Then $m_{\boldsymbol{\lambda}} g_w g_i =$

$$\begin{cases} m_{\boldsymbol{\lambda}} g_{ws_i} \text{ if } \ell(ws_i) > \ell(w) \\ m_{\boldsymbol{\lambda}} g_{ws_i} \text{ if } \ell(ws_i) < \ell(w) \text{ and } i, i+1 \text{ are in different blocks of } (A_{\boldsymbol{\lambda}})w \\ m_{\boldsymbol{\lambda}}(g_{ws_i} + (q-q^{-1})m_{\boldsymbol{\lambda}} g_w) \text{ if } \ell(ws_i) < \ell(w) \text{ and } i, i+1 \text{ are in the same block of } (A_{\boldsymbol{\lambda}})w. \end{cases}$$

PROOF. Suppose that $\ell(ws_i) > \ell(w)$ and let $s_{j_1} \cdots s_{j_k}$ be a reduced expression for w. Then $s_{j_1} \cdots s_{j_k} s_i$ is a reduced expression for ws_i and so $g_{ws_i} = g_w g_{s_i}$ by definition. On the other hand, if $\ell(ws_i) < \ell(w)$ then w has a reduced expression ending in s_i, therefore

$$g_w g_i = g_{ws_i} g_i^2 = g_{ws_i}(1+(q-q^{-1})e_i g_i) = g_{ws_i} + (q-q^{-1}) g_w e_i.$$

On the other hand, from Lemma 2 we have that $E_{A_{\boldsymbol{\lambda}}} g_w e_i = g_w E_{A_{\boldsymbol{\lambda}} w} e_i$ which is equal to $g_w E_{A_{\boldsymbol{\lambda}} w}$ or zero depending on whether i and $i+1$ are in the same block of $A_{\boldsymbol{\lambda}}$ or not. This concludes the proof of the Lemma. □

With these preparations, we are in position to give the definition of the set of elements that turn out to contain the cellular basis for $\mathcal{Y}_{r,n}(q)$.

Definition 24 Let $\boldsymbol{\lambda} \in Comp_{r,n}$ and suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard multitableaux of shape $\boldsymbol{\lambda}$. Then we define

$$m_{\mathfrak{s}\mathfrak{t}} := g^*_{d(\mathfrak{s})} m_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})}. \tag{57}$$

In particular we have $m_{\boldsymbol{\lambda}} = m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$.

Clearly we have $m^*_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{t}\mathfrak{s}}$, as one sees from the definition of $*$.

Our goal is to show that with $\mathfrak{s}$ and $\mathfrak{s}$ running over standard multitableaux for multipartitions, the $m_{\mathfrak{s}\mathfrak{t}}$'s form a cellular basis for $\mathcal{Y}_{r,n}(q)$. A first property of $m_{\mathfrak{s}\mathfrak{t}}$ is given by the following Lemma.

Lemma 25 Suppose that $\boldsymbol{\lambda} \in Comp_{r,n}$ and that $\mathfrak{s}$ and $\mathfrak{t}$ are $\boldsymbol{\lambda}$-multitableaux. If i and j occur in the same component of $\mathfrak{t}$ then we have that $m_{\mathfrak{s}\mathfrak{t}} e_{ij} = m_{\mathfrak{s}\mathfrak{t}}$. Otherwise $m_{\mathfrak{s}\mathfrak{t}} e_{ij} = 0$. A similar statement holds for $e_{ij} m_{\mathfrak{s}\mathfrak{t}}$.

PROOF. From the definitions we have

$$m_{\mathfrak{s}\mathfrak{t}} e_{ij} = g^*_{d(\mathfrak{s})} x_{\boldsymbol{\lambda}} U_{\boldsymbol{\lambda}} E_{A_{\boldsymbol{\lambda}}} g_{d(\mathfrak{t})} e_{ij} = g^*_{d(\mathfrak{s})} x_{\boldsymbol{\lambda}} U_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})} E_{A_{\boldsymbol{\lambda}} d(\mathfrak{t})} e_{ij}.$$

The elements of the blocks of $A_{\boldsymbol{\lambda}} d(\mathfrak{t})$ are exactly the elements of the components of $\mathfrak{t}$ and so the Lemma follows from the definition of E_A. The case $e_{ij} m_{\mathfrak{s}\mathfrak{t}}$ is treated similarly or by applying $*$ to the first case. □

Lemma 26 Let $\boldsymbol{\lambda} \in Comp_{r,n}$ and let $\mathfrak{s}$ and $\mathfrak{t}$ be row standard $\boldsymbol{\lambda}$-multitableaux. Then for $h \in \mathcal{Y}_{r,n}(q)$ we have that $m_{\mathfrak{s}\mathfrak{t}}h$ is a linear combination of terms of the form $m_{\mathfrak{s}\mathfrak{v}}$ where $\mathfrak{v}$ is a row standard $\boldsymbol{\lambda}$-multitableau. A similar statement holds for $hm_{\mathfrak{s}\mathfrak{t}}$.

PROOF. Using Lemma 23 we get that $m_{\mathfrak{s}\mathfrak{t}}h$ a linear combination of terms of the form $m_{\mathfrak{s}\mathfrak{t}^{\lambda}}g_w$. For each such w we find a $y \in \mathfrak{S}_{\boldsymbol{\lambda}}$ and a distinguished right coset representative d of $\mathfrak{S}_{\boldsymbol{\lambda}}$ in $\mathfrak{S}_n$ such that $w = yd$ and $\ell(w) = \ell(y) + \ell(d)$. Hence, via Lemma 23 we get that

$$m_{\mathfrak{s}\mathfrak{t}}h = q^{\ell(y)} m_{\mathfrak{s}\mathfrak{t}^{\lambda}} g_d = q^{\ell(y)} m_{\mathfrak{s}\mathfrak{v}}$$

where $\mathfrak{v} = \mathfrak{t}^{\boldsymbol{\lambda}} g_d$ is row standard. This proves the Lemma in the case $m_{\mathfrak{s}\mathfrak{t}}h$. The case $hm_{\mathfrak{s}\mathfrak{t}}$ is treated similarly or by applying $*$ to the first case.

□

The proof of the next Lemma is inspired by the proof of Proposition 3.18 of Dipper, James and Mathas' paper [11], although it should be noted that the basic setup of [11] is different from ours. Just like in that paper, our proof relies on Murphy's Theorem 4.18 in [34], which is a key ingredient for the construction of the standard basis for $\mathcal{H}_n(q)$.

Lemma 27 Suppose that $\boldsymbol{\lambda} \in Comp_{r,n}$ and that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard $\boldsymbol{\lambda}$-multitableaux. Then there are multipartitions $\boldsymbol{\mu} \in Par_{r,n}$ and standard multitableaux $\mathfrak{u}$ and $\mathfrak{v}$ of shape $\boldsymbol{\mu}$, such that $\mathfrak{u} \trianglerighteq \mathfrak{s}$, $\mathfrak{v} \trianglerighteq \mathfrak{t}$ and such that $m_{\mathfrak{s}\mathfrak{t}}$ is a linear combination of the corresponding elements $m_{\mathfrak{u}\mathfrak{v}}$.

PROOF. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$. Let us first consider the case where only one of the components $\lambda^{(i)}$ is nonempty. Then $A_{\boldsymbol{\lambda}} = \mathbf{n}$ and using Lemma 21, we have that

$$m_{\mathfrak{s}\mathfrak{t}} = g^*_{d(\mathfrak{s})} m_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})} = g_{d(\mathfrak{s})^{-1}} U_{\boldsymbol{\lambda}} E_{\mathbf{n}} x_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})} = U_{\boldsymbol{\lambda}} E_{\mathbf{n}} g_{d(\mathfrak{s})^{-1}} x_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})} = U_{\boldsymbol{\lambda}} E_{\mathbf{n}} g^*_{d(\mathfrak{s})} x_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})}.$$

We then use Murphy's result Theorem 4.18 of [34] on $g^*_{d(\mathfrak{s})} x_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})}$ and get that it can be written as a linear combination of elements

$$g^*_{d(\mathfrak{u})} x_{\boldsymbol{\mu}} g_{d(\mathfrak{v})} = m_{\mathfrak{u}\mathfrak{v}}$$

where $\boldsymbol{\mu} \in Par_{1,n}$ is a usual partition and $\mathfrak{u}$ and $\mathfrak{v}$ are standard $\boldsymbol{\mu}$-tableaux satisfying $\mathfrak{u} \trianglerighteq \mathfrak{s}$ and $\mathfrak{v} \trianglerighteq \mathfrak{t}$. Since $U_{\boldsymbol{\lambda}} = U_{\boldsymbol{\mu}}$ and $E_{A_{\boldsymbol{\lambda}}} = E_{A_{\boldsymbol{\mu}}}$ we then get via Lemma 21 that $U_{\boldsymbol{\lambda}} E_{A_{\boldsymbol{\lambda}}} g^*_{d(\mathfrak{s})} x_{\boldsymbol{\lambda}} g_{d(\mathfrak{t})}$ is a linear combination of elements

$$U_{\boldsymbol{\mu}} E_{A_{\boldsymbol{\mu}}} g^*_{d(\mathfrak{u})} x_{\boldsymbol{\mu}} g_{d(\mathfrak{v})} = g^*_{d(\mathfrak{u})} E_{A_{\boldsymbol{\lambda}}} U_{\boldsymbol{\mu}} x_{\boldsymbol{\mu}} g_{d(\mathfrak{v})} = m_{\mathfrak{u}\mathfrak{v}}$$

where $\mathfrak{u}$ and and $\mathfrak{v}$ are standard $\boldsymbol{\mu}$-tableaux satisfying $\mathfrak{u} \trianglerighteq \mathfrak{s}$ and $\mathfrak{v} \trianglerighteq \mathfrak{t}$, as claimed.

Let us now consider the general case in which more than one of the $\lambda^{(i)}$'s are nonempty. Let α be the composition $\alpha = (\alpha_1, \dots, \alpha_s) := \mathrm{red}\|\boldsymbol{\lambda}\|$ with corresponding Young subgroup $\mathfrak{S}_\alpha = \mathfrak{S}_{\mathrm{red}\,\alpha} = \mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_s}$. There exist $\boldsymbol{\lambda}$-multitableaux $\mathfrak{s}_0$ and $\mathfrak{t}_0$ of the initial kind together with $w_{\mathfrak{s}}, w_{\mathfrak{t}} \in \mathfrak{S}_n$ such that $d(\mathfrak{s}) = d(\mathfrak{s}_0) w_{\mathfrak{s}}$, $d(\mathfrak{t}) = d(\mathfrak{t}_0) w_{\mathfrak{t}}$ and $\ell(d(\mathfrak{s})) = \ell(d(\mathfrak{s}_0)) + \ell(w_{\mathfrak{s}})$ and $\ell(d(\mathfrak{t})) = \ell(d(\mathfrak{t}_0)) + \ell(w_{\mathfrak{t}})$. Thus, $w_{\mathfrak{s}}$ and $w_{\mathfrak{t}}$ are distinguished right coset representatives for $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$ and using Lemma 23, together with its left action version obtained via $*$, we get that $m_{\mathfrak{s}\mathfrak{t}} = g^*_{w_{\mathfrak{s}}} m_{\mathfrak{s}_0\mathfrak{t}_0} g_{w_{\mathfrak{t}}}$. But both $E_{A_{\boldsymbol{\lambda}}}$ and $U_{\boldsymbol{\lambda}}$ have decompositions that correspond to the one for $\mathfrak{S}_\alpha$ and hence we have

$$m_{\mathfrak{s}_0\mathfrak{t}_0} = m_{\mathfrak{s}_0^{(1)}\mathfrak{t}_0^{(1)}} m_{\mathfrak{s}_0^{(2)}\mathfrak{t}_0^{(2)}} \dots m_{\mathfrak{s}_0^{(r)}\mathfrak{t}_0^{(s)}} \tag{58}$$

where $\mathfrak{s}_0^{(i)}$ and $\mathfrak{t}_0^{(i)}$ are row standard tableaux on the numbers permuted by $\mathfrak{S}_{\alpha_i}$. On each of the factors $m_{\mathfrak{s}_0^{(i)}\mathfrak{t}_0^{(i)}}$ we now apply the result of the first part of the proof, thus

concluding that $m_{\mathfrak{s}_0^{(i)}\mathfrak{t}_0^{(i)}}$ is a linear combination of terms of the form $m_{\mathfrak{u}_0^{(i)}\mathfrak{v}_0^{(i)}}$ where $\mathfrak{u}_0^{(i)}$ and $\mathfrak{v}_0^{(i)}$ are standard $\mu_0^{(i)}$-tableaux on the same numbers as $\mathfrak{s}_0^{(i)}$ and $\mathfrak{t}_0^{(i)}$ where $\mu_0^{(i)}$ is now a partition, and satisfying $\mathfrak{u}_0^{(i)} \trianglerighteq \mathfrak{s}_0^{(i)}$ and $\mathfrak{v}_0^{(i)} \trianglerighteq \mathfrak{t}_0^{(i)}$. But then, with the same reasoning that was used for (58), but on the product of each of these terms, we get that $m_{\mathfrak{s}_0\mathfrak{t}_0}$ is a linear combination of $m_{\mathfrak{u}_0\mathfrak{v}_0}$ where $\mathfrak{u}_0$ and $\mathfrak{v}_0$ are standard multitableaux for multipartitions $\boldsymbol{\mu}$ where $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$. Hence $m_{\mathfrak{s}\mathfrak{t}} = g^*_{w_\mathfrak{s}} m_{\mathfrak{s}_0\mathfrak{t}_0} g_{w_\mathfrak{t}}$ is a linear combination of terms $g^*_{w_\mathfrak{s}} m_{\mathfrak{u}_0\mathfrak{v}_0} g_{w_\mathfrak{t}}$, On the other hand we have that $g^*_{w_\mathfrak{s}} m_{\mathfrak{u}_0\mathfrak{v}_0} g_{w_\mathfrak{t}} = g^*_{w_\mathfrak{s}} g^*_{d(\mathfrak{u}_0)} m_{\boldsymbol{\mu}} g_{d(\mathfrak{v}_0)} g_{w_\mathfrak{t}} = g^*_{d(\mathfrak{u}_0)w_\mathfrak{s}} m_{\boldsymbol{\mu}} g_{d(\mathfrak{v}_0)w_\mathfrak{t}} = m_{\mathfrak{u}_0 w_\mathfrak{s}, \mathfrak{v}_0 w_\mathfrak{t}}$ where we used that $\alpha = \|\boldsymbol{\mu}\|$ together with the fact that $w_\mathfrak{s}$ and $w_\mathfrak{t}$ are distinguished right coset representatives for $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$. The Lemma now follows. □

Corollary 28 Suppose that $\boldsymbol{\lambda} \in Comp_{r,n}$ and that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard $\boldsymbol{\lambda}$-multitableaux. If $h \in \mathcal{Y}_{r,n}(q)$, then $m_{\mathfrak{s}\mathfrak{t}}h$ is a linear combination of terms of the form $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard $\boldsymbol{\mu}$-multitableaux for some multipartition $\boldsymbol{\mu} \in Par_{r,n}$ and $\mathfrak{u} \trianglerighteq \mathfrak{s}$ and $\mathfrak{v} \trianglerighteq \mathfrak{t}$. A similar statement holds for $hm_{\mathfrak{s}\mathfrak{t}}$.

PROOF. This is now immediate from the Lemmas 26 and 27. □

So far our construction of the cellular basis has followed the layout used in [11], with the appropriate adaptions. But to show that the $m_{\mathfrak{s}\mathfrak{t}}$'s generate $\mathcal{Y}_{r,n}(q)$ we shall deviate from that path. We turn our attention to the R-subalgebra $\mathcal{T}_n$ of $\mathcal{Y}_{r,n}$ generated by $t_1, t_2, \ldots, t_n$. By the faithfulness of $V^{\otimes n}$, it is isomorphic to the subalgebra $\mathfrak{T}_n \subset \mathrm{End}(V^{\otimes n})$ considered above. Our proof that the elements $m_{\mathfrak{s}\mathfrak{t}}$ generate $\mathcal{Y}_{r,n}(q)$ relies on the, maybe surprising, fact that $\mathcal{T}_n$ is compatible with the $\{m_{\mathfrak{s}\mathfrak{t}}\}$, in the sense that the elements of $\{m_{\mathfrak{s}\mathfrak{t}}\}$ that correspond to pairs of standard multitableaux of one-column multipartitions induce a basis for $\mathcal{T}_n$.

As already mentioned, we consider our $m_{\mathfrak{s}\mathfrak{t}}$ as the natural generalization of Murphy's standard basis to $\mathcal{Y}_{r,n}(q)$. It is interesting to note that Murphy's standard basis and its generalization have already before manifested 'good' compatibility properties of the above kind.

Let us first define a one-column r-multipartition to an element of $Par_{r,n}$ of the form $((1^{c_1}), \ldots, (1^{c_r}))$ and let $Par^1_{r,n}$ be the set of one-column r-multipartitions. Note that there is an obvious bijection between $Par^1_{r,n}$ and the set of usual compositions in r parts. We define

$$\mathrm{Std}^1_{n,r} := \left\{\mathfrak{s} \,|\, \mathfrak{s} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in Par^1_{r,n}\right\}.$$

Note that $\mathrm{Std}^1_{n,r}$ has cardinality r^n as follows from the multinomial formula.

Lemma 29 For all $\mathfrak{s} \in \mathrm{Std}^1_{n,r}$, we have that $m_{\mathfrak{s}\mathfrak{s}}$ belongs to $\mathcal{T}_n$.

PROOF. Let $\mathfrak{s}$ be an element of $\mathrm{Std}^1_{n,r}$. It general, it is useful to think of $d(\mathfrak{s}) \in \mathfrak{S}_n$ as the row reading of $\mathfrak{s}$, that is the element obtained by reading the components of $\mathfrak{s}$ from left to right, and the rows of each component from top to bottom.

We show by induction on $\ell(d(\mathfrak{s}))$ that $m_{\mathfrak{s}\mathfrak{s}}$ belongs to $\mathcal{T}_n$. If $\ell(d(\mathfrak{s})) = 0$ then $x_{\boldsymbol{\lambda}} = 1$ and so $m_{\mathfrak{s}\mathfrak{s}} = U_{\boldsymbol{\lambda}} E_{A_{\boldsymbol{\lambda}}}$ that certainly belongs to $\mathcal{T}_n$. Assume that the statement holds for all multitableaux $\mathfrak{s}' \in \mathrm{Std}^1_{n,r}$ such that $\ell(d(\mathfrak{s}')) < \ell(d(\mathfrak{s}))$. Choose i such that i occurs in $\mathfrak{s}$ to the right of $i+1$: such an i exists because $\ell(d(\mathfrak{s})) \neq 0$. Then we can apply the inductive hypothesis to $\mathfrak{s}s_i$, that is $m_{\mathfrak{s}s_i\,\mathfrak{s}s_i} \in \mathcal{T}_n$. But then

$$m_{\mathfrak{s}\mathfrak{s}} = g^*_{d(\mathfrak{s})} m_{\boldsymbol{\lambda}} g_{d(\mathfrak{s})} = g_i m_{\mathfrak{s}s_i\,\mathfrak{s}s_i} g_i = g_i m_{\mathfrak{s}s_i\,\mathfrak{s}s_i} (g_i^{-1} + (q - q^{-1}) e_i). \tag{59}$$

But $g_i m_{\mathfrak{s}s_i\,\mathfrak{s}s_i} g_i^{-1}$ certainly belongs to $\mathcal{T}_n$, as one sees from relation (6). Finally, from Lemma 25 we get that $m_{\mathfrak{s}s_i\,\mathfrak{s}s_i} e_i = 0$, thus proving the Lemma. □

Lemma 30 Suppose that $\boldsymbol{\lambda} \in Comp_{r,n}$ and let $\mathfrak{s}$ and $\mathfrak{t}$ be $\boldsymbol{\lambda}$-multitableaux. Then for all $k = 1, \dots, n$ we have that

$$m_{\mathfrak{t}\mathfrak{s}} t_k = \xi^{p_{\mathfrak{s}}(k)} m_{\mathfrak{t}\mathfrak{s}} \text{ and } t_k m_{\mathfrak{t}\mathfrak{s}} = \xi^{p_{\mathfrak{t}}(k)} m_{\mathfrak{t}\mathfrak{s}}.$$

PROOF. From (6) we have that $g_w t_k = t_{kw^{-1}} g_w$ for all $w \in \mathfrak{S}_n$. Then, by Lemma 21(2) we have

$$m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{s}} t_k = m_{\boldsymbol{\lambda}} g_{d(\mathfrak{s})} t_k = m_{\boldsymbol{\lambda}} g_{d(\mathfrak{s})} t_k = m_{\boldsymbol{\lambda}} t_{kd(\mathfrak{s})^{-1}} g_{d(\mathfrak{s})} = \xi^{p_{\boldsymbol{\lambda}}(kd(\mathfrak{s})^{-1})} m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{s}}.$$

On the other hand, since $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}} d(\mathfrak{s})$ we have that $p_{\boldsymbol{\lambda}}(kd(\mathfrak{s})^{-1}) = p_{\mathfrak{s}}(k)$ and hence $m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{s}} t_k = \xi^{p_{\mathfrak{s}}(k)} m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{s}}$. Multiplying this equality on the left by $g^*_{d(\mathfrak{t})}$, the proof of the first formula is completed. The second formula is shown similarly or by applying $*$ to the first. □

Our next Proposition shows that the set $\{m_{\mathfrak{s}\mathfrak{s}}\}$, where $\mathfrak{s} \in \mathrm{Std}^1_{n,r}$, forms a basis for $\mathcal{T}_n$, as promised. We already know that $m_{\mathfrak{s}\mathfrak{s}} \in \mathcal{T}_n$ and that the cardinality of $\mathrm{Std}^1_{n,r}$ is r^n which is the dimension of $\mathcal{T}_n$, but even so the result is not completely obvious, since we are working over the ground ring R which is not a field.

Proposition 31 $\{m_{\mathfrak{s}\mathfrak{s}} \,|\, \mathfrak{s} \in \mathrm{Std}^1_{n,r}\}$ is an R-basis for $\mathcal{T}_n$.

PROOF. Recall that we showed in the proof of Theorem 10 that

$$V_{i_1,i_2\dots,i_n} = \mathrm{Span}_R\{v^{j_1}_{i_1} \otimes v^{j_2}_{i_2} \otimes \cdots \otimes v^{j_n}_{i_n} \,|\, j_k \in \mathbb{Z}/r\mathbb{Z}\}$$

is a faithful $\mathcal{T}_n$-module for any fixed, but arbitrary, set of lower indices. Let seq_n be the set of sequences $\underline{i} = (i_1, i_2, \dots, i_n)$ of numbers $1 \le i_j \le n$. Then we have that

$$V^{\otimes n} = \bigoplus_{\underline{i} \in \mathrm{seq}_n} V_{\underline{i}} \tag{60}$$

and of course $V^{\otimes n}$ is a faithful $\mathcal{T}_n$-module, too. For $\mathfrak{s} \in \mathrm{Std}^1_{n,r}$ and $\underline{i} \in \mathrm{seq}_n$ we define

$$v^{\mathfrak{s}}_{\underline{i}} := v^{j_1}_{i_1} \otimes v^{j_2}_{i_2} \otimes \ldots \otimes v^{j_n}_{i_n} \in V_{\underline{i}} \tag{61}$$

where $(j_1, j_2, \dots j_n) := (p_{\mathfrak{s}}(1), p_{\mathfrak{s}}(2), \dots p_{\mathfrak{s}}(n))$. Then $\{v^{\mathfrak{s}}_{\underline{i}} \,|\, \mathfrak{s} \in \mathrm{Std}^1_{n,r}, \underline{i} \in \mathrm{seq}_n\}$ is an R-basis for $V^{\otimes n}$. We now claim the following formula in $V_{\underline{i}}$:

$$v^{\mathfrak{t}}_{\underline{i}} m_{\mathfrak{s}\mathfrak{s}} = \begin{cases} v^{\mathfrak{t}}_{\underline{i}} & \text{if } \mathfrak{s} = \mathfrak{t} \\ 0 & \text{otherwise.} \end{cases} \tag{62}$$

We show it by induction on $\ell(d(\mathfrak{s}))$. If $\ell(d(\mathfrak{s})) = 0$, then $\mathfrak{s} = t^{\boldsymbol{\lambda}}$ where $\boldsymbol{\lambda}$ is the shape of $\mathfrak{s}$. We have $x_{\boldsymbol{\lambda}} = 1$ and so $m_{\mathfrak{s}\mathfrak{s}} = m_{\boldsymbol{\lambda}} = U_{\boldsymbol{\lambda}} E_{A_{\boldsymbol{\lambda}}}$. We then get (62) directly from the definitions of $U_{\boldsymbol{\lambda}}$ and $E_{A_{\boldsymbol{\lambda}}}$ together with Lemma 6.

Let now $\ell(d(\mathfrak{s})) \neq 0$ and assume that (62) holds for multitableaux $\mathfrak{s}'$ such that $\ell(d(\mathfrak{s}')) < \ell(d(\mathfrak{s}))$. We choose j such that j occurs in $\mathfrak{s}$ to the right of $j+1$. Using (59) we have that $m_{\mathfrak{s}\mathfrak{s}} = g_j m_{\mathfrak{s}s_j \mathfrak{s}s_j} g_j^{-1}$. On the other hand, j and $j+1$ occur in different components of $\mathfrak{s}$ and so by Definition 5 of the $\mathcal{Y}_{r,n}(q)$-action in $V^{\otimes n}$ we get that $v^{\mathfrak{s}}_{\underline{i}} g_j^{\pm 1} = v^{\mathfrak{s}s_j}_{\underline{i}s_j}$, corresponding to the first case of (34). Hence we get via the inductive hypothesis that

$$v^{\mathfrak{s}}_{\underline{i}} m_{\mathfrak{s}\mathfrak{s}} = v^{\mathfrak{s}}_{\underline{i}} g_j m_{\mathfrak{s}s_j \mathfrak{s}s_j} g_j^{-1} = v^{\mathfrak{s}s_j}_{\underline{i}s_j} m_{\mathfrak{s}s_j \mathfrak{s}s_j} g_j^{-1} = v^{\mathfrak{s}s_j}_{\underline{i}s_j} g_j^{-1} = v^{\mathfrak{s}}_{\underline{i}}$$

which shows the first part of (62).

If $\mathfrak{s} \neq \mathfrak{t}$ then we essentially argue the same way. We choose j as before and may apply the inductive hypothesis to $\mathfrak{s}s_j$. We have that $v_{\underline{i}}^{\mathfrak{t}} m_{\mathfrak{s}\mathfrak{s}} = v_{\underline{i}}^{\mathfrak{t}} g_j m_{\mathfrak{s}s_j\,\mathfrak{s}s_j} g_j^{-1}$ and so need to determine $v_{\underline{i}}^{\mathfrak{t}} g_j$. This is slightly more complicated than in the first case, but using the Definition 5 of the $\mathcal{Y}_{r,n}(q)$-action in $V^{\otimes n}$ we get that $v_{\underline{i}}^{\mathfrak{t}} g_j$ is always an R-linear combination of the vectors $v_{\underline{i}s_j}^{\mathfrak{t}s_j}$ and $v_{\underline{i}}^{\mathfrak{t}s_j}$: indeed in the cases $s = t$ of Definition 5 we have that $p_{\mathfrak{t}s_j}(s) = p_{\mathfrak{t}}(s)$. But $\mathfrak{s} \neq \mathfrak{t}$ implies that $\mathfrak{s}s_j \neq \mathfrak{t}s_j$ and so we get by the inductive hypothesis that

$$v_{\underline{i}}^{\mathfrak{t}} m_{\mathfrak{s}\mathfrak{s}} = v_{\underline{i}}^{\mathfrak{t}} g_j m_{\mathfrak{s}s_j\,\mathfrak{s}s_j} g_j^{-1} = 0 \tag{63}$$

and (62) is proved.

From (62) we now deduce that $\sum_{\mathfrak{s}\in \mathrm{Std}_{n,r}^1} v_{\underline{i}}^{\mathfrak{t}} m_{\mathfrak{s}\mathfrak{s}} = v_{\underline{i}}^{\mathfrak{t}}$ for any $\mathfrak{t}$ and $\underline{i}$, and hence

$$\sum_{\mathfrak{s}\in \mathrm{Std}_{n,r}^1} m_{\mathfrak{s}\mathfrak{s}} = 1 \tag{64}$$

since $V^{\otimes n}$ is faithful and the $\{v_{\underline{i}}^{\mathfrak{t}}\}$ form a basis for $V^{\otimes n}$. We then get that

$$t_i = t_i 1 = \sum_{\mathfrak{s}\in \mathrm{Std}_{n,r}^1} t_i m_{\mathfrak{s}\mathfrak{s}} = \sum_{\mathfrak{s}\in \mathrm{Std}_{n,r}^1} \xi^{p_{\mathfrak{s}}(i)} m_{\mathfrak{s}\mathfrak{s}} \tag{65}$$

and hence, indeed, the set $\{m_{\mathfrak{s}\mathfrak{s}} \mid \mathfrak{s} \in \mathrm{Std}_{n,r}^1\}$ generates $\mathcal{T}_n$. On the other hand, the R-independence of $\{m_{\mathfrak{s}\mathfrak{s}}\}$ follows easily from (62), via evaluation on the vectors $v_{\underline{i}}^{\mathfrak{t}}$. The Theorem is proved. □

Theorem 32 The algebra $\mathcal{Y}_{r,n}(q)$ is a free R-module with basis

$$\mathcal{B}_{r,n} = \left\{ m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for some multipartition } \boldsymbol{\lambda} \text{ of } n \right\}.$$

Moreover, $(\mathcal{B}_{r,n}, Par_{r,n})$ is a cellular basis of $\mathcal{Y}_{r,n}(q)$ in the sense of Definition 18.

Proof. From Proposition 31, we have that 1 is an R-linear combination of elements $m_{\mathfrak{s}\mathfrak{s}}$ where $\mathfrak{s}$ are certain standard multitableaux. Thus, via Corollary 28 we get that $\mathcal{B}_{r,n}$ spans $\mathcal{Y}_{r,n}(q)$. On the other hand, the cardinality of $\mathcal{B}_{r,n}$ is $r^n n!$ since, for example, $\mathcal{B}_{r,n}$ is the set of tableaux for the Ariki-Koike algebra whose dimension is $r^n n!$. But this implies that $\mathcal{B}_{r,n}$ is an R-basis for $\mathcal{Y}_{r,n}(q)$. Indeed, from Juyumaya's basis we know that $\mathcal{Y}_{r,n}(q)$ has rank $N := r^n n!$ and any surjective homomorphism $f : R^N \mapsto R^N$ splits since R^N is projective.

The multiplicative property that $\mathcal{B}_{r,n}$ must satisfy in order to be a cellular basis of $\mathcal{Y}_{r,n}(q)$, can now be shown by repeating the argument of Proposition 3.25 of [11]. For the reader's convenience, we sketch the argument.

Let first $\mathcal{Y}_{r,n}^{\boldsymbol{\lambda}}(q)$ be the R-submodule of $\mathcal{Y}_{r,n}(q)$ spanned by

$$\{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\mu}) \text{ for some } \boldsymbol{\mu} \in Par_{r,n} \text{ and } \boldsymbol{\mu} \rhd \boldsymbol{\lambda}\}.$$

Then one checks using Lemma 27 that $\mathcal{Y}_{r,n}^{\boldsymbol{\lambda}}$ is an ideal of $\mathcal{Y}_{r,n}(q)$. Using Lemma 27 once again, we get for $h \in \mathcal{Y}_{r,n}(q)$ the formula

$$m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} h = \sum_{\mathfrak{v}} r_{\mathfrak{v}} m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} \mod \mathcal{Y}_{r,n}^{\boldsymbol{\lambda}}$$

where $r_{\mathfrak{v}} \in R$. This is so because $\mathfrak{t}^{\boldsymbol{\lambda}}$ is a maximal element of $\mathrm{Std}(\boldsymbol{\lambda})$. Multiplying this equation on the left with $g^*_{d(\mathfrak{s})}$ we get the formula

$$m_{\mathfrak{s}\mathfrak{t}}h = \sum_{\mathfrak{v}} r_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{v}} \mod \mathcal{Y}^{\boldsymbol{\lambda}}_{r,n}$$

and this is the multiplicative property that is required for cellularity. □

As already explained in [14], the existence of a cellular basis in an algebra A has strong consequences for the modular representation theory of A. Here we give an application of our cellular basis $\mathcal{B}_{r,n}$ that goes in a somewhat different direction, obtaining from it Lusztig's idempotent presentation of $\mathcal{Y}_{r,n}(q)$, used in [27], [28].

Proposition 33 The Yokonuma-Hecke algebra $\mathcal{Y}_{r,n}(q)$ is isomorphic to the associative R-algebra generated by the elements $\{g_i | i = 1,\dots,n-1\}$ and $\{f_{\mathfrak{s}} | \mathfrak{s} \in \mathrm{Std}^1_{n,r}\}$ subject to the following relations:

$$g_i g_j = g_j g_i \qquad \text{for } |i-j|>1 \tag{66}$$
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \qquad \text{for all } i=1,\dots,n-2 \tag{67}$$
$$f_{\mathfrak{s}} g_i = g_i f_{\mathfrak{s}s_i} \qquad \text{for all } \mathfrak{s}, i \tag{68}$$
$$g_i^2 = 1 + (q-q^{-1}) \sum_{\mathfrak{s}\in \mathrm{Std}^1_{n,r}} \delta_{i,i+1}(\mathfrak{s}) f_{\mathfrak{s}} g_i \qquad \text{for all } i \tag{69}$$
$$\sum_{\mathfrak{s}\in \mathrm{Std}^1_{n,r}} f_{\mathfrak{s}} = 1 \qquad \text{for all } \mathfrak{s} \tag{70}$$
$$f_{\mathfrak{s}} f_{\mathfrak{s}'} = \delta_{\mathfrak{s},\mathfrak{s}'} f_{\mathfrak{s}} \qquad \text{for all } \mathfrak{s},\mathfrak{s}' \in \mathrm{Std}^1_{n,r} \tag{71}$$

where $\delta_{\mathfrak{s},\mathfrak{s}'}$ is the Kronecker delta function on $\mathrm{Std}^1_{n,r}$ and where we set $\delta_{i,i+1}(\mathfrak{s}) := 1$ if i and $i+1$ belong to the same component (column) of $\mathfrak{s}$, otherwise $\delta_{i,i+1}(\mathfrak{s}) := 0$. Moreover, we define $f_{\mathfrak{s}s_i} := f_{\mathfrak{s}}$ if $\delta_{i,i+1}(\mathfrak{s}) = 0$.

PROOF. Let $\mathcal{Y}'_{r,n}$ be the R-algebra defined by the presentation of the Lemma. Then there is an R-algebra homomorphism $\varphi : \mathcal{Y}'_{r,n} \to \mathcal{Y}_{r,n}(q)$, given by $\varphi(g_i) := g_i$ and $\varphi(f_{\mathfrak{s}}) := m_{\mathfrak{s}\mathfrak{s}}$. Indeed, the $m_{\mathfrak{s}\mathfrak{s}}$'s are orthogonal idempotents and have sum 1 as we see from (62) and (64) respectively. Moreover, using (59), (62) and (64) we get that the relations (68), (69), (70) and (71) hold with $m_{\mathfrak{s}\mathfrak{s}}$ replacing $f_{\mathfrak{s}}$, and finally the first two relations hold trivially.

On the other hand, using (65) we get that φ is a surjection and since $\mathcal{Y}'_{r,n}$ is generated over R by the set $\{g_w f_{\mathfrak{s}} | w \in \mathfrak{S}_n, \mathfrak{s} \in \mathrm{Std}^1_{n,r}\}$ of cardinality $r^n n!$, we get that φ is also an injection. □

Remark 34 The relations given in the Proposition are the relations, for type A, of the algebra H_n considered in 31.2 of [27] see also [31]. We would like to draw the attention to the sum appearing in the quadratic relation (69), making it look rather different than the quadratic relation of Yokonuma's or Juyumaya's presentation. In 31.2 of [27], it is mentioned that H_n is closely related to the convolution algebra associated with a Chevalley group and its unipotent radical and indeed in 35.3 of [28], elements of this algebra are found that satisfy the relations of H_n. However, we could not find a Theorem in *loc. cit.*, stating explicitly that H_n is isomorphic to $\mathcal{Y}_{r,n}(q)$.

5. JUCYS-MURPHY ELEMENTS

In this section we show that the Jucys-Murphy elements J_i for $\mathcal{Y}_{r,n}(q)$, introduced by Chlouveraki and Poulain d'Andecy in [8], are JM-elements in the abstract sense de-

fined by Mathas, see [33]. This is with respect to the cellular basis for $\mathcal{Y}_{r,n}(q)$ obtained in the previous section.

We first consider the elements J'_k of $\mathcal{Y}_{r,n}(q)$ given by $J'_1 = 0$ and for $k \geq 1$

$$J'_{k+1} = q^{-1}(e_k g_{(k,k+1)} + e_{k-1,k+1} g_{(k-1,k+1)} + \cdots + e_{1,k+1} g_{(1,k+1)}) \tag{72}$$

where $g_{(i,k+1)}$ is g_w for $w = (i, k+1)$. These elements are generalizations of the Jucys-Murphy elements for the Iwahori-Hecke algebra $\mathcal{H}_n(q)$, in the sense that we have $E_{\mathbf{n}} J'_k = E_{\mathbf{n}} L_k$, where L_k are the Jucys-Murphy elements for $\mathcal{H}_n(q)$ defined in [32].

The elements J_i of $\mathcal{Y}_{r,n}(q)$ that we shall refer to as Jucys-Murphy elements were introduced by Chlouveraki and Poulain d'Andecy in [8] via the recursion

$$J_1 = 1 \quad \text{and} \quad J_{i+1} = g_i J_i g_i \quad \text{for } i = 1, \ldots, n-1. \tag{73}$$

The relation between J_i and J'_i is given by

$$J_i = 1 + (q^2 - 1) J'_i. \tag{74}$$

In fact, in [8] the elements $\{J_1, \ldots, J_n\}$, as well as the elements $\{t_1, \ldots, t_n\}$, are called Jucys-Murphy elements for the Yokonuma-Hecke algebra.

The following definition appears for the first time in [33]. It formalizes the concept of Jucys-Murphy elements.

Definition 35 Suppose that the $\mathcal{R}$-algebra A is cellular with antiautomorphism $*$ and cellular basis $\mathcal{C} = \{a_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Lambda, \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$. Suppose moreover that each set $T(\lambda)$ is endowed with a poset structure with order relation $\rhd_\lambda$. Then we say that the a commuting set $\mathcal{L} = \{L_1, \ldots, L_M\} \subseteq A$ is a family of JM-elements for A, with respect to the basis $\mathcal{C}$, if it satisfies that $L_i^* = L_i$ for all i and if there exists a set of scalars $\{c_{\mathfrak{t}}(i) \mid \mathfrak{t} \in T(\lambda),\ 1 \leq i \leq M\}$, called the contents of λ, such that for all $\lambda \in \Lambda$ and $\mathfrak{t} \in T(\lambda)$ we have that

$$a_{\mathfrak{s}\mathfrak{t}} L_i = c_{\mathfrak{t}}(i) a_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{v} \in T(\lambda) \\ \mathfrak{v} \rhd_\lambda \mathfrak{t}}} r_{\mathfrak{s}\mathfrak{v}} a_{\mathfrak{s}\mathfrak{v}} \mod A^\lambda \tag{75}$$

for some $r_{\mathfrak{s}\mathfrak{v}} \in \mathcal{K}$.

Our goal is to prove that the set

$$\mathcal{L}_{\mathcal{Y}_{r,n}} := \{L_1, \ldots, L_{2n} \mid L_k = J_k,\ L_{n+k} = t_k,\ 1 \leq k \leq n\} \tag{76}$$

is a family of JM-elements for $\mathcal{Y}_{r,n}(q)$ in the above sense. Let us start out by stating the following Lemma.

Lemma 36 Let i and k be integers such that $1 \leq i < n$ and $1 \leq k \leq n$. Then

(1) g_i and J_k commute if $i \neq k-1, k$.

(2) $\mathcal{L}_{\mathcal{Y}_{r,n}}$ is a set of commuting elements.

(3) g_i commutes with $J_i J_{i+1}$ and $J_i + J_{i+1}$.

(4) $g_i J_i = J_{i+1} g_i + (q^{-1} - q) e_i J_{i+1}$ and $g_i J_{i+1} = J_i g_i + (q - q^{-1}) e_i J_{i+1}$.

PROOF. For the proof of (1) and (2), see [8, Corollaries 1 and 2]. We then prove (3) using (1) and (2) and induction on i. For $i = 1$ the two statements are trivial. For $i > 1$ we have that

$$\begin{aligned} g_i J_i J_{i+1} &= g_i (g_{i-1} J_{i-1} g_{i-1})(g_i g_{i-1} J_{i-1} g_{i-1} g_i) = g_i g_{i-1} J_{i-1} g_i g_{i-1} g_i J_{i-1} g_{i-1} g_i \\ &= g_i g_{i-1} g_i J_{i-1} g_{i-1} g_i J_{i-1} g_{i-1} g_i = g_{i-1} (g_i g_{i-1} J_{i-1} g_{i-1} g_i) J_{i-1} g_{i-1} g_i \\ &= g_{i-1} J_{i+1} J_{i-1} g_{i-1} g_i = (g_{i-1} J_{i-1} g_{i-1}) J_{i+1} g_i = J_i J_{i+1} g_i \end{aligned}$$

and

$$
\begin{aligned}
g_i(J_i+J_{i+1}) &= g_iJ_i + g_i^2 J_i g_i = g_iJ_i + (1+(q-q^{-1})e_ig_i)J_ig_i \\
&= J_ig_i + g_iJ_i(1+(q-q^{-1})e_ig_i) = J_ig_i + g_iJ_ig_i^2 = (J_i+J_{i+1})g_i.
\end{aligned}
$$

Finally, the equalities of (4) are shown by direct computations, that we leave to the reader.

□

Let $\mathcal{K}$ be an R-algebra as above, such that $q \in \mathcal{K}^\times$. Let $\mathfrak{t}$ be a $\boldsymbol{\lambda}$-multitableau and suppose that the node of $\mathfrak{t}$ labelled by (x,y,k) is filled in with j. Then we define the *quantum content* of j as the element $c_{\mathfrak{t}}(j) := q^{2(y-x)} \in \mathcal{K}$. We furthermore define $\mathrm{res}_{\mathfrak{t}}(j) := y-x$ and then have the formula $c_{\mathfrak{t}}(j) = q^{2\mathrm{res}_{\mathfrak{t}}(j)}$. When $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$, we write $c_{\boldsymbol{\lambda}}(j)$ for $c_{\mathfrak{t}}(j)$.

The next Proposition is the main result of this section.

Proposition 37 $(\mathcal{Y}_{r,n}(q), \mathcal{B}_{r,n})$ is a cellular algebra with family of JM-elements $\mathcal{L}_{\mathcal{Y}_{r,n}}$ and contents given by

$$
d_{\mathfrak{t}}(k) := \begin{cases} c_{\mathfrak{t}}(k) & \text{if } k = 1,\dots,n \\ \xi_{p_{\mathfrak{t}}(k)} & \text{if } k = n+1,\dots,2n. \end{cases}
$$

PROOF. We have already proved that $\mathcal{B}_{r,n}$ is a cellular basis for $\mathcal{Y}_{r,n}(q)$, so we only need to prove that the elements of $\mathcal{L}_{\mathcal{Y}_{r,n}}$ verify the conditions of Definition 35.

For the order $\rhd_{\boldsymbol{\lambda}}$ on $\mathrm{Std}(\boldsymbol{\lambda})$ we shall use the dominance order $\rhd$ on multitableaux that was introduced above. By Lemma 30 the JM-condition (75) holds for $k = n+1,\dots,2n$ and so we only need to check the cases $k = 1,\dots,n$.

Let us first consider the case when $\mathfrak{t}$ is a standard $\boldsymbol{\lambda}$-multitableau of the initial kind. Set $\alpha = \|\boldsymbol{\lambda}\|$, with corresponding Young subgroup $\mathfrak{S}_\alpha = \mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_r}$ and suppose that k belongs to the l'th component of $\mathfrak{t}$. Now, by (1) of Lemma 36 we get that

$$
\begin{gathered}
m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} J_k = m_{\mathfrak{t}^{\lambda^{(1)}}\mathfrak{t}^{(1)}} \cdots m_{\mathfrak{t}^{\lambda^{(l)}}\mathfrak{t}^{(l)}} J_k m_{\mathfrak{t}^{\lambda^{(l+1)}}\mathfrak{t}^{(l+1)}} \cdots m_{\mathfrak{t}^{\lambda^{(r)}}\mathfrak{t}^{(r)}} = \\
m_{\mathfrak{t}^{\lambda^{(1)}}\mathfrak{t}^{(1)}} \cdots m_{\mathfrak{t}^{\lambda^{(l)}}\mathfrak{t}^{(l)}} (1+(q^2-1)J_k') m_{\mathfrak{t}^{\lambda^{(l+1)}}\mathfrak{t}^{(l+1)}} \cdots m_{\mathfrak{t}^{\lambda^{(r)}}\mathfrak{t}^{(r)}}
\end{gathered}
$$

where the $\mathfrak{t}^{\lambda^{(i)}}$'s and $\mathfrak{t}^{(i)}$'s are the components of $\mathfrak{t}^{\boldsymbol{\lambda}}$ and $\mathfrak{t}$. On the other hand, using Lemma 25 together with the definition of J_i' we have that

$$
m_{\mathfrak{t}^{\lambda^{(l)}}\mathfrak{t}^{(l)}}(1+(q^2-1)J_k') = m_{\mathfrak{t}^{\lambda^{(l)}}\mathfrak{t}^{(l)}}(1+(q^2-1)L_k^l)
$$

where $L_k^l = q^{-1}(g_{(k,k+1)} + g_{(k-1,k+1)} + \cdots + g_{(m,k+1)})$ is the k'th Jucys-Murphy element as in [32] for the Iwahori-Hecke algebra corresponding to $\mathfrak{S}_{\alpha_l}$. Applying [32, Theorem 3.32] on this factor we get that $m_{\mathfrak{t}^{\lambda^{(l)}}\mathfrak{t}^{(l)}}(1+(q^2-1)L_k^l)$ is equal to

$$
\begin{aligned}
& m_{\mathfrak{t}^{\lambda^{(l)}}\mathfrak{t}^{(l)}} + (q^2-1)[\mathrm{res}_{\mathfrak{t}^{(l)}}(k)]_q m_{\mathfrak{t}^{\lambda^{(l)}}\mathfrak{t}^{(l)}} + \sum_{\substack{\mathfrak{v} \in \mathrm{Std}(\lambda^{(l)}) \\ \mathfrak{v} \rhd \mathfrak{t}^{(l)}}} a_{\mathfrak{v}} m_{\mathfrak{t}^{\lambda^{(l)}}\mathfrak{v}} + \sum_{\substack{\mathfrak{a}_1,\mathfrak{b}_1 \in \mathrm{Std}(\mu^{(l)}) \\ \mu^{(l)} \rhd \lambda^{(l)}}} r_{\mathfrak{a}_1\mathfrak{b}_1} m_{\mathfrak{a}_1\mathfrak{b}_1} \\
&= q^{2(\mathrm{res}_{\mathfrak{t}^{(l)}}(k))} m_{\mathfrak{t}^{\lambda^{(l)}}\mathfrak{t}^{(l)}} + \sum_{\substack{\mathfrak{v} \in \mathrm{Std}(\lambda^{(l)}) \\ \mathfrak{v} \rhd \mathfrak{t}^{(l)}}} a_{\mathfrak{v}} m_{\mathfrak{t}^{\lambda^{(l)}}\mathfrak{v}} + \sum_{\substack{\mathfrak{a}_1,\mathfrak{b}_1 \in \mathrm{Std}(\mu^{(l)}) \\ \mu^{(l)} \rhd \lambda^{(l)}}} r_{\mathfrak{a}_1\mathfrak{b}_1} m_{\mathfrak{a}_1\mathfrak{b}_1}
\end{aligned}
\tag{77}
$$

for some $r_{\mathfrak{a}_1\mathfrak{b}_1}, a_{\mathfrak{v}} \in R$ where the tableaux $\mathfrak{a}_1, \mathfrak{b}_1 \in \mathrm{Std}(\mu^{(l)})$ involve the numbers permuted by $\mathfrak{S}_{\alpha_l}$. For $\mathfrak{s}, \mathfrak{t}$ and $\mathfrak{v}$ appearing in the sum set $\mathfrak{a} := (\mathfrak{t}^{\lambda^{(1)}}, \dots, \mathfrak{a}_1, \dots, \mathfrak{t}^{\lambda^{(r)}})$, $\mathfrak{b} := (\mathfrak{t}^{(1)}, \dots, \mathfrak{b}_1, \dots, \mathfrak{t}^{(r)})$ and $\mathfrak{c} := (\mathfrak{t}^{\lambda^{(1)}}, \dots, \mathfrak{v}, \dots, \mathfrak{t}^{\lambda^{(r)}})$. Then $\mathfrak{c} \in \mathrm{Std}(\boldsymbol{\lambda})$, $\mathfrak{a}, \mathfrak{b} \in \mathrm{Std}(\boldsymbol{\mu})$ where

$\boldsymbol{\mu} := (\lambda^{(1)}, \dots, \mu^{(l)}, \dots \lambda^{(r)})$. Moreover, by our definition of the dominance order we have $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$, $\mathfrak{c} \rhd \mathfrak{t}$ and so $m_{\mathfrak{a}\mathfrak{b}} \in \mathcal{Y}_{r,n}^{\boldsymbol{\lambda}}$. On the other hand, we have

$$m_{\mathfrak{t}^{\lambda^{(1)}}\mathfrak{t}^{(1)}} \cdots m_{\mathfrak{a}_1\mathfrak{b}_1} \cdots m_{\mathfrak{t}^{\lambda^{(r)}}\mathfrak{t}^{(r)}} = g^*_{d(\mathfrak{a}_1)} m_{\mathfrak{t}^{\lambda^{(1)}}\mathfrak{t}^{(1)}} \cdots m_{\mu^{(l)}\mu^{(l)}} \cdots m_{\mathfrak{t}^{\lambda^{(r)}}\mathfrak{t}^{(r)}} g_{d(\mathfrak{b}_1)} = m_{\mathfrak{a}\mathfrak{b}}$$

Multiplying (77) on the left with $m_{\mathfrak{t}^{\lambda^{(1)}}\mathfrak{t}^{(1)}} \cdots$ and on the right with $\cdots m_{\mathfrak{t}^{\lambda^{(r)}}\mathfrak{t}^{(r)}}$ and using $\mathrm{res}_{\mathfrak{t}^{(l)}}(k) = \mathrm{res}_{\mathfrak{t}}(k)$ we then get

$$m_{\mathfrak{t}} J_k \quad = \quad c_{\mathfrak{t}}(k) m_{\mathfrak{t}} + \sum_{\substack{\mathfrak{c} \in \mathrm{Std}(\boldsymbol{\lambda}) \\ \mathfrak{c} \rhd \mathfrak{t}}} a_{\mathfrak{c}} m_{\mathfrak{c}}$$

modulo $\mathcal{Y}_{r,n}^{\boldsymbol{\lambda}}$ which shows the Proposition for $\mathfrak{t}$ of the initial kind.

For $\mathfrak{t}$ a general multitableau, there exists a multitableau $\mathfrak{t}_0$ of the initial kind together with a distinguished right coset representative $w_{\mathfrak{t}}$ of $\mathfrak{S}_\alpha$ in $\mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{t}_0 w_{\mathfrak{t}}$. Let $w_{\mathfrak{t}} = s_{i_1} s_{i_2} \dots s_{i_k}$ be a reduced expression for $w_{\mathfrak{t}}$. Then we have that i_j and $i_j + 1$ are located in different blocks of $\mathfrak{t}_0 s_{i_1} \dots s_{i_{j-1}}$ for all $j \geq 1$ and that $\mathfrak{t}_0 s_{i_1} \dots s_{i_{j-1}} s_{i_j}$ is obtained from $\mathfrak{t}_1 s_{i_1} \dots s_{i_{j-1}}$ by interchanging i_j and $i_j + 1$. Using Lemma 25 and (4) of Lemma 36 we now get that

$$m_{\mathfrak{t}^\lambda \mathfrak{t}} J_k = m_{\mathfrak{t}^\lambda \mathfrak{t}_0} g_{w_{\mathfrak{t}}} J_k = m_{\mathfrak{t}^\lambda \mathfrak{t}_0} J_{k w_{\mathfrak{t}}^{-1}} g_{w_{\mathfrak{t}}}.$$

Since $\mathfrak{t}_0$ is of the initial kind, we get

$$\begin{aligned} m_{\mathfrak{t}} J_k \quad &= m_{\mathfrak{t}_0} J_{k w_{\mathfrak{t}}^{-1}} g_{w_{\mathfrak{t}}} = \left(c_{\mathfrak{t}_0}(k w_{\mathfrak{t}}^{-1}) m_{\mathfrak{t}_0} + \sum_{\substack{\mathfrak{v}_0 \in \mathrm{Std}(\lambda) \\ \mathfrak{v}_0 \rhd \mathfrak{t}_0}} a_{\mathfrak{v}_0} m_{\mathfrak{v}_0} \right) g_{w_{\mathfrak{t}}} \\ &= c_{\mathfrak{t}}(k) m_{\mathfrak{t}} + \sum_{\substack{\mathfrak{v} \in \mathrm{Std}(\lambda) \\ \mathfrak{v} \rhd \mathfrak{t}}} a_{\mathfrak{v}} m_{\mathfrak{v}} \end{aligned}$$

where we used that the occurring $\mathfrak{v}_0$ are all of the initial kind such that $m_{\mathfrak{v}} = m_{\mathfrak{v}_0} g_{w_{\mathfrak{t}}}$ with $\mathfrak{v} \rhd \mathfrak{t}_0$ and $a_{\mathfrak{v}} = a_{\mathfrak{v}_0}$. This finishes the proof of the Proposition. □

In view of the Proposition, we can now apply the general theory developped in [33]. In particular, we recover the semisimplicity criterium of Chlouveraki and Poulain d'Andecy, [8], and can even generalize it to the case of ground fields of positive characteristic. We leave the details to the reader.

6. Representation theory of the algebra of braids and ties.

In this final section we once again turn our attention to the algebra $\mathcal{E}_n(q)$ of braids and ties.

In the paper [36], the representation theory of $\mathcal{E}_n(q)$ was studied in the generic case, where a parametrizing set for the irreducible modules was found. On the other hand, the dimensions of the generically irreducible modules were not determined in that paper. In this section we show that $\mathcal{E}_n(q)$ is a cellular algebra by giving a concrete combinatorial construction of a cellular basis for it. As a bonus we obtain a closed formula for the dimensions of the cell modules, which in particular gives a formula for the irreducible modules in the generic case. Although the construction of the cellular basis for $\mathcal{E}_n(q)$ follows the outline of the construction of the cellular basis $\mathcal{B}_{r,n}$ for $\mathcal{Y}_{r,n}(q)$, the combinatorial details are quite a lot more involved and, as we shall see, involve a couple of new ideas.

Recall that $\mathcal{E}_n(q)$ is the $S := \mathbb{Z}[q, q^{-1}]$-algebra defined by the generators and relations given in Definition 3 and that it was shown in [36] that $\mathcal{E}_n(q)$ has an S-basis of

the form $\{E_A g_w | A \in \mathcal{SP}_n, w \in \mathfrak{S}_n\}$. (Theorem 14 gave a new proof of this basis). We shall often need the following relations in $\mathcal{E}_n(q)$, that have already appeared implicitly above

$$E_A g_w = g_w E_{Aw} \text{ and } E_A E_B = E_C \text{ for } w \in \mathfrak{S}_n, A, B \in \mathcal{SP}_n \tag{78}$$

where $C \in \mathcal{SP}_n$ is minimal with respect to $A \subseteq C, B \subseteq C$.

6.1. Decomposition of $\mathcal{E}_n(q)$

In this subsection we obtain central idempotents of $\mathcal{E}_n(q)$ and a corresponding subalgebra decomposition of $\mathcal{E}_n(q)$. This is inspired by I. Marin's recent paper [29], which in turn is inspired by [40] and [15].

Recall that for a finite poset $(\Gamma, \preceq)$ there is an associated Möebius function $\mu_\Gamma : \Gamma \times \Gamma \to \mathbb{Z}$. In our set partition case $(\mathcal{SP}_n, \subseteq)$ the Möebius function $\mu_{\mathcal{SP}_n}$ is given by the formula

$$\mu_{\mathcal{SP}_n}(A,B) = \begin{cases} (-1)^{r-s} \prod_{i=1}^{r-1} (i!)^{r_{i+1}} & \text{if } A \subseteq B \\ 0 & \text{otherwise} \end{cases} \tag{79}$$

where r and s are the number of blocks of A and B respectively, and where r_i is the number of blocks of B containing exactly i blocks of A.

We use the Möebius function $\mu = \mu_{\mathcal{SP}_n}$ to introduce a set of orthogonal idempotents elements of $\mathcal{E}_n(q)$. This is a special case of the general construction given in *loc. cit.* For $A \in \mathcal{SP}_n$ the idempotent $\mathbb{E}_A \in \mathcal{E}_n(q)$ is given by the formula

$$\mathbb{E}_A := \sum_{A \subseteq B} \mu(A,B) E_B. \tag{80}$$

For example, we have $\mathbb{E}_{\{\{1\},\{2\},\{3\}\}} = E_{\{\{1\},\{2\},\{3\}\}} - E_{\{\{1,2\},\{3\}\}} - E_{\{\{1\},\{2,3\}\}} - E_{\{\{1,3\},\{2\}\}} + 2E_{\{\{1,2,3\}\}}$. We have the following result.

Proposition 38 The following properties hold.

(1) $\{\mathbb{E}_A | A \in \mathcal{SP}_n\}$ is a set of orthogonal idempotents of $\mathcal{E}_n(q)$.

(2) For all $w \in \mathfrak{S}_n$ and $A \in \mathcal{SP}_n$ we have $\mathbb{E}_A g_w = g_w \mathbb{E}_{Aw}$.

(3) For all $A \in \mathcal{SP}_n$ we have $\mathbb{E}_A E_B = \begin{cases} \mathbb{E}_A & \text{if } B \subseteq A \\ 0 & \text{if } B \not\subseteq A. \end{cases}$

PROOF. We have already mentioned (1) so let us prove (2). We first note that the order relation $\subseteq$ on $\mathcal{SP}_n$ is compatible with the action of $\mathfrak{S}_n$ on $\mathcal{SP}_n$ that is $A \subseteq B$ if and only if $Aw \subseteq Bw$ for all $w \in \mathfrak{S}_n$. This implies that $\mu(Aw, Bw) = \mu(A,B)$ for all $w \in \mathfrak{S}_n$. From this we get, via (78), that

$$\mathbb{E}_A g_w = g_w \sum_{A \subseteq B} \mu(A,B) E_{Bw} = g_w \sum_{A \subseteq Cw^{-1}} \mu(A, Cw^{-1}) E_C = g_w \sum_{Aw \subseteq C} \mu(Aw, C) E_C = g_w \mathbb{E}_{Aw}$$

showing (2). Finally, we obtain (3) from the orthogonality of the $\mathbb{E}_A$'s and the formula $E_A = \mathbb{E}_A + \sum_{A \subset B} \lambda_B \mathbb{E}_B$, which is obtained by inverting (80). □

We say that a set partition of $\mathbf{n}$, $A = \{I_{i_1}, \ldots, I_{i_k}\}$, has type $\alpha \in \mathcal{P}ar_n$ if there exists a permutation σ such that $(|I_{i_{(1)\sigma}}|, \ldots, |I_{i_{(k)\sigma}}|) = \alpha$. For example, the set partitions of $\mathbf{3}$ having type $(2,1)$ are $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$ and $\{\{2,3\},\{1\}\}$. For short, we write $|A| = \alpha$ if $A \in \mathcal{SP}_n$ has type α.

For each $\alpha \in \mathcal{P}ar_n$ we define the following element $\mathbb{E}_\alpha$ of $\mathcal{E}_n(q)$

$$\mathbb{E}_\alpha := \sum_{|A| = \alpha} \mathbb{E}_A. \tag{81}$$

Then by Proposition 38 we have that $\{\mathbb{E}_\alpha | \alpha \in \mathcal{P}ar_n\}$ is a set of central orthogonal idempotents of $\mathcal{E}_n(q)$, which is complete: $\sum_{\alpha\in\mathcal{P}ar_n}\mathbb{E}_\alpha = 1$. As an immediate consequence we get the following decomposition of $\mathcal{E}_n(q)$ into a direct sum of two-sided ideals

$$\mathcal{E}_n(q) = \bigoplus_{\alpha\in\mathcal{P}ar_n} \mathcal{E}_n^\alpha(q) \tag{82}$$

where we define $\mathcal{E}_n^\alpha(q) := \mathbb{E}_\alpha \mathcal{E}_n(q)$.

Using the $\{E_A g_w\}$-basis for $\mathcal{E}_n(q)$, together with part (3) of Proposition 38, we get that the set

$$\{\mathbb{E}_A g_w \mid w \in \mathfrak{S}_n, |A| = \alpha\} \tag{83}$$

is an S-basis for $\mathcal{E}_n^\alpha(q)$. In particular, we have that the dimension of $\mathcal{E}_n^\alpha(q)$ is $b_n(\alpha)n!$, where $b_n(\alpha)$ is the number of set partitions of $\mathbf{n}$ having type $\alpha \in \mathcal{P}ar_n$. The numbers $b_n(\alpha)$ are the socalled Faà di Bruno coefficients and are given by the following formula

$$b_n(\alpha) = \frac{n!}{(k_1!)^{m_1} m_1! \cdots (k_r!)^{m_r} m_r!} \tag{84}$$

where $\alpha = (k_1^{m_1}, \dots, k_r^{m_r})$ and $k_1 > \dots > k_r$.

6.2. Cellular basis for $\mathcal{E}_n(q)$

Let us explain the ingredients of our cellular basis for $\mathcal{E}_n(q)$. The antiautomorphism $*$ is easy to explain, since one easily checks on the relations for $\mathcal{E}_n(q)$ that $\mathcal{E}_n(q)$ is endowed with an S-linear antiautomorphism $*$, satisfying $e_i^* := e_i$ and $g_i^* := g_i$. We have that $\mathbb{E}_A^* = \mathbb{E}_A$.

Next we explain the poset denoted Λ in Definition 18 of cellular algebras. By general principles, it should be the parametrizing set for the irreducible modules for $\mathcal{E}_n(q)$ in the generic situation, so let us therefore recall this set $\mathcal{L}_n$ from [36]. $\mathcal{L}_n$ is the set of pairs $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ where $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ is an m-multipartition of n. We require that $\boldsymbol{\lambda}$ be *increasing* by which we mean that $\lambda^{(i)} < \lambda^{(j)}$ only if $i < j$ where $<$ is any fixed extension of the usual dominance order on partitions to a total order, and where we set $\lambda < \mu$ if λ and τ are partitions such that $|\lambda| < |\tau|$.

In order to describe the $\boldsymbol{\mu}$-ingredient of Λ we need to introduce some more notation. The multiplicities of equal $\lambda^{(i)}$'s give rise to a composition of m. To be more precise, let m_1 be the maximal i such that $\lambda^{(1)} = \lambda^{(2)} = \dots = \lambda^{(i)}$, let m_2 be the maximal i such that $\lambda^{(m_1+1)} = \lambda^{(m_1+2)} = \dots = \lambda^{(m_1+i)}$, and so on until m_q. Then we have that $m = m_1 + \dots + m_q$. We then require that $\boldsymbol{\mu}$ be of the form $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(q)})$ where each $\mu^{(i)}$ is partition of m_i. This is our description of $\mathcal{L}_n$ as a set.

We now need to describe $\mathcal{L}_n$ as a poset. Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ and $\overline{\Lambda} = (\overline{\boldsymbol{\lambda}} \mid \overline{\boldsymbol{\mu}})$ are elements of $\mathcal{L}_n$ such that $\|\boldsymbol{\lambda}\| = \|\overline{\boldsymbol{\lambda}}\|$. We first write $\boldsymbol{\lambda} \rhd_1 \overline{\boldsymbol{\lambda}}$ if $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ and $\overline{\boldsymbol{\lambda}} = (\overline{\lambda^{(1)}}, \dots, \overline{\lambda^{(m)}})$ and if there exists a permutation σ such that $(\lambda^{(1\sigma)}, \dots, \lambda^{(m\sigma)}) \rhd (\overline{\lambda^{(1)}}, \dots, \overline{\lambda^{(m)}})$ where $\rhd$ is the dominance order on m-multipartitions, introduced above. We then say that $\Lambda \rhd \overline{\Lambda}$ if $\boldsymbol{\lambda} \rhd_1 \overline{\boldsymbol{\lambda}}$ or if $\boldsymbol{\lambda} = \overline{\boldsymbol{\lambda}}$ and $\mu^{(i)} \rhd \overline{\mu^{(i)}}$ for all i. As usual we set $\Lambda \unrhd \overline{\Lambda}$ if $\Lambda \rhd \overline{\Lambda}$ or if $\Lambda = \overline{\Lambda}$. This is our description of $\mathcal{L}_n$ as a poset. Note that if $\|\boldsymbol{\lambda}\| \neq \|\overline{\boldsymbol{\lambda}}\|$ then Λ and $\overline{\Lambda}$ are by definition not comparable.

Remark 39 We could have introduced an order '$\succ$' on $\mathcal{L}_n$ by replacing '$\rhd_1$' by '$\rhd$' in the above definition, that is $\Lambda \succ \overline{\Lambda}$ if $\boldsymbol{\lambda} \rhd \overline{\boldsymbol{\lambda}}$ or if $\boldsymbol{\lambda} = \overline{\boldsymbol{\lambda}}$ and $\mu^{(i)} \rhd \overline{\mu^{(i)}}$ for all i. Then '$\succ$' is a finer order than '$\rhd$', but in general they are different. The reason why we need to

work with '$\rhd$' rather than '$\succ$' comes from the straightening procedure of Lemma 51 below.

We could also have introduced an order on $\mathcal{L}_n$ by replacing '=' with '$=_1$' in the above definition, where '$=_1$' is defined via a permutation σ, similar to what we did for $\rhd_1$: that is $\Lambda \succ \overline{\Lambda}$ if $\boldsymbol{\lambda} \rhd \overline{\boldsymbol{\lambda}}$ or if $\boldsymbol{\lambda} =_1 \overline{\boldsymbol{\lambda}}$ and $\mu^{(i)} \rhd \overline{\mu^{(i)}}$ for all i. On the other hand, since $\boldsymbol{\lambda}$ and $\overline{\boldsymbol{\lambda}}$ are assumed to be increasing multipartitions, we get that '$=_1$' is just usual equality '=' and hence we would get the same order on $\mathcal{L}_n$.

Let us give an example to illustrate our order.

Example 40 We first note that $(3,3,1) \rhd (3,2,2)$ in the dominance order on partitions, but both are incomparable with the partition $(4,1,1,1)$. Suppose now that $(3,2,2) < (4,1,1,1) < (3,3,1)$ in our extension of the dominance order. We then consider the following increasing multipartitions of 25

$$\boldsymbol{\lambda} = ((2),(2),(3,2,2),(4,1,1,1),(3,3,1)) \text{ and } \overline{\boldsymbol{\lambda}} = ((2),(2),(3,2,2),(3,2,2),(4,1,1,1)).$$

Then we have that $\boldsymbol{\lambda}$ and $\overline{\boldsymbol{\lambda}}$ are increasing multipartitions, but incomparable in the dominance order on multipartitions. On the other hand $\boldsymbol{\lambda} \rhd_1 \overline{\boldsymbol{\lambda}}$ via the permutation $\sigma = s_4$ and hence we have the following relation in $\mathcal{L}_n$

$$\Lambda := \Big(\boldsymbol{\lambda} \mid \big((2),(1),(1),(1)\big)\Big) \rhd \Big(\overline{\boldsymbol{\lambda}} \mid \Big(\big((1^2),(2),(1)\big)\Big) =: \overline{\Lambda}.$$

For $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n$ as above, we next define the concept of Λ-tableaux. Suppose that $\mathbb{t}$ is a pair $\mathbb{t} = (\mathbf{t} \mid \mathbf{u})$. Then $\mathbb{t}$ is called a Λ-tableau if $\mathbf{t} = (t^{(1)}, \ldots, t^{(m)})$ is a multitableau of n in the usual sense, satisfying $Shape(t^{(i)}) = \lambda^{(i)}$, and $\mathbf{u}$ is of the form $\mathbf{u} = (u_1, \ldots, u_q)$ where each u_i is a tableau of shape $\mu^{(i)}$ in the usual sense. As usual, if $\mathbb{t}$ is Λ-tableau we define $Shape(\mathbb{t}) := \Lambda$.

Let $\mathrm{Tab}(\Lambda)$ denote the set of Λ-tableaux and let $\mathrm{Tab}_n := \cup_{\Lambda \in \mathcal{L}_n} \mathrm{Tab}(\Lambda)$. We then say that $\mathbb{t} = (\mathbf{t} \mid \mathbf{u}) \in \mathrm{Tab}(\Lambda)$ is row standard if all its ingredients are row standard multitableaux in the usual sense.

We say that $\mathbb{t} = (\mathbf{t} \mid \mathbf{u}) \in \mathrm{Tab}(\Lambda)$ is standard if all its ingredients are standard (multi)-tableaux and if moreover $\mathbf{t}$ is an *increasing* multitableau. By increasing we here mean that whenever $\lambda^{(i)} = \lambda^{(j)}$ we have that $i < j$ if and only if $\min(t^{(i)}) < \min(t^{(j)})$ where $\min(t)$ is the function that reads off the minimal entry of the tableau t. We define $\mathrm{Std}(\Lambda)$ to be the set of all standard Λ-tableaux.

Example 41 For $\Lambda = \Big(\big((1,1),(2),(2),(2,1)\big) \mid \big((1),(1,1),(1)\big)\Big)$ we consider the following Λ-tableaux

$$\begin{aligned} \mathbb{t}_1 &:= \left(\left(\begin{array}{|c|}\hline 1 \\ \hline 9 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 5 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 6 & 8 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 7 \\ \cline{1-1}\end{array} \right) \mid \left(\begin{array}{|c|}\hline 1 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline\end{array} \right) \right) \\ \mathbb{t}_2 &:= \left(\left(\begin{array}{|c|}\hline 1 \\ \hline 9 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 5 & 6 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 8 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 7 \\ \cline{1-1}\end{array} \right) \mid \left(\begin{array}{|c|}\hline 1 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline\end{array} \right) \right). \end{aligned} \tag{85}$$

Then by our definition, $\mathbb{t}_1$ is a standard Λ-tableau, but $\mathbb{t}_2$ is not.

Remark 42 The use of the function $\min(\cdot)$ is somewhat arbitrary. In fact we could have used any injective function with values in a totally ordered set.

For $\mathbf{t} = (t^{(1)}, \ldots, t^{(m)})$ and $\overline{\mathbf{t}} = (\overline{t^{(1)}}, \ldots, \overline{t^{(m)}})$ we define $\mathbf{t} \rhd_1 \overline{\mathbf{t}}$ if there exists a permutation σ such that $(t^{(1\sigma)}, \ldots, t^{(m\sigma)}) \rhd (\overline{t^{(1)}}, \ldots, \overline{t^{(m)}})$ in the sense of multitableaux. We then extend the order on $\mathcal{L}_n$ to Tab_n as follows. Suppose that $\mathbb{t} = (\mathbf{t} \mid \mathbf{u}) \in \mathrm{Tab}(\Lambda)$ and $\overline{\mathbb{t}} = (\overline{\mathbf{t}} \mid \overline{\mathbf{u}}) \in \mathrm{Tab}(\overline{\Lambda})$ and that $\Lambda \trianglerighteq \overline{\Lambda}$. Then we say that $\mathbf{t} \rhd \overline{\mathbf{t}}$ if $\mathbf{t} \rhd_1 \overline{\mathbf{t}}$ or if $\mathbf{t} = \overline{\mathbf{t}}$ and

$u^{(i)} \rhd \overline{u}^{(i)}$ for all i. As usual we set $\mathfrak{t} \trianglerighteq \overline{\mathfrak{t}}$ if $\mathfrak{t} \rhd \overline{\mathfrak{t}}$ or $\mathfrak{t} = \overline{\mathfrak{t}}$. This finishes our description of Λ-tableaux as a poset.

Suppose that $\alpha \in \mathcal{P}ar_n$. In the sequel we say that an increasing multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ has type α if $\|\boldsymbol{\lambda}\|^{\mathrm{op}} := (|\lambda^{(m)}|, \dots, |\lambda^{(1)}|) = \alpha$.

From the basis of $\mathcal{E}_n(q)$ mentioned above, we have that $\dim \mathcal{E}_n(q) = b_n n!$ where b_n is the n'th Bell number, that is the number of set partitions on $\mathbf{n}$. Our next Lemma is a first strong indication of the relationship between our notion of standard tableaux and the representation theory of $\mathcal{E}_n(q)$.

Recall the notation $d_\lambda := |\mathrm{Std}(\lambda)|$ that we introduced for partitions λ. In the proof of the Lemma, and later on, we shall use repeatedly the formula $\sum_{\lambda \in \mathcal{P}ar_n} d_\lambda^2 = n!$.

Lemma 43 With the above notation we have that $\sum_{\Lambda \in \mathcal{L}_n} |\mathrm{Std}(\Lambda)|^2 = b_n n!$.

Proof. For $\alpha \in \mathcal{P}ar_n$ we first define $\mathcal{L}_n(\alpha) := \{(\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n \big| \|\boldsymbol{\lambda}\|^{\mathrm{op}} = \alpha\}$. Then it is enough to prove the formula

$$\sum_{\Lambda \in \mathcal{L}_n(\alpha)} |\mathrm{Std}(\Lambda)|^2 = b_n(\alpha) n! \tag{86}$$

where $b_n(\alpha)$ is the Faà di Bruno coefficient introduced above. Let us first consider the case $\alpha = (k^m)$, that is $n = mk$. Then we have

$$b_n(m,k) := b_n(\alpha) = \frac{1}{m!}\binom{n}{k \cdots k}$$

with k appearing m times in the multinomial coefficient. Let $\{\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(d)}\}$ be the fixed ordered enumeration of all the partitions of k. If $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$ then $\boldsymbol{\lambda}$ has the form

$$\boldsymbol{\lambda} = (\overbrace{\lambda^{(1)}, \dots, \lambda^{(1)}}^{m_1}, \overbrace{\lambda^{(2)}, \dots, \lambda^{(2)}}^{m_2}, \dots, \overbrace{\lambda^{(d)} \dots, \lambda^{(d)}}^{m_d})$$

where the m_i's are non-negative integers with sum m and $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(d)})$ is a multipartition of type $\|\boldsymbol{\mu}\| = (m_1, m_2, \dots, m_d)$. The number of increasing multitableaux of shape $\boldsymbol{\lambda}$ is

$$\frac{1}{m_1! \dots m_d!}\binom{n}{k \cdots k} \prod_{j=1}^{d} d_{\lambda^{(j)}}^{m_j}$$

whereas the number of standard tableaux of shape $\boldsymbol{\mu}$ is $\prod_{j=1}^{d} d_{\mu^{(j)}}$ and so we get

$$|\mathrm{Std}(\Lambda)| = \frac{1}{m!}\binom{m}{m_1 \cdots m_d}\binom{n}{k \cdots k} \prod_{j=1}^{d} d_{\lambda^{(j)}}^{m_j} d_{\mu^{(j)}} \tag{87}$$

By first fixing $\boldsymbol{\lambda}$ and then letting each $\mu^{(i)}$ vary over all possibilities we get that the square sum of the above $|\mathrm{Std}(\Lambda)|$'s is the sum of

$$\binom{n}{k \cdots k}^2 \prod_{j=1}^{d} \frac{d_{\lambda^{(j)}}^{2m_j}}{m_j!} = \binom{n}{k \cdots k}^2 \frac{1}{m!}\binom{m}{m_1 \cdots m_d} \prod_{j=1}^{d} d_{\lambda^{(j)}}^{2m_j}$$

with the m_i's running over the above mentioned set of numbers. But by the multinomial formula, this sum is equal to

$$\binom{n}{k \cdots k}^2 \frac{1}{m!}\left(\sum_{j=1}^{d} d_{\lambda^{(j)}}^2\right)^m = \binom{n}{k \cdots k}^2 \frac{1}{m!} k!^m = \frac{n!}{m!}\binom{n}{k \cdots k} = b_n(\alpha) n!$$

and (86) is proved in this case.

Let us now consider the general case where $\alpha = (k_1^{M_1}, \dots, k_r^{M_r})$, where $k_1 > \dots > k_r$. Set $n_i = k_i M_i$, $M := M_1 + \ldots + M_r$. Then $n = n_1 + \ldots + n_r$ and the Faà di Bruno coefficient $b_n(\alpha)$ is given by the formula

$$b_n(\alpha) = \binom{n}{n_1 \cdots n_r} b_{n_1}(M_1, k_1) \cdots b_{n_r}(M_r, k_r). \tag{88}$$

Let us now consider the square sum $\sum_{\Lambda \in \mathcal{L}_n(\alpha)} |\mathrm{Std}(\Lambda)|^2$. For $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n(\alpha)$ we split $\boldsymbol{\lambda}$ into multipartitions $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_r$, where $\boldsymbol{\lambda}_1 = (\lambda^{(1)}, \dots, \lambda^{(M_1)})$, $\boldsymbol{\lambda}_2 = (\lambda^{(M_1+1)}, \dots, \lambda^{(M_1+M_2)})$, and so on. We split $\boldsymbol{\mu}$ correspondingly into $\boldsymbol{\mu}_i$'s and set $\Lambda_i := (\boldsymbol{\lambda}_i \mid \boldsymbol{\mu}_i)$. Then $\Lambda_i \in \mathcal{L}_{n_i}((k_i^{M_i}))$ and we have

$$|\mathrm{Std}(\Lambda)| = \binom{n}{n_1 \cdots n_r} |\mathrm{Std}_{n_1}(\Lambda_1)| \cdots |\mathrm{Std}_{n_r}(\Lambda_r)| \tag{89}$$

where $\mathrm{Std}_{n_i}(\Lambda_i)$ means standard tableaux of shape Λ_i on $\mathbf{n_i}$. Combining (86), (88) and (89) we get that

$$\sum_{\Lambda \in \mathcal{L}_n(\alpha)} |\mathrm{Std}(\Lambda)|^2 = n! b_n(\alpha)$$

as claimed. □

Corollary 44 Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n$ is above with $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(q)})$ and set $n_i := |\lambda^{(i)}|$ and $m_i := |\mu^{(i)}|$. Then we have that

$$|\mathrm{Std}(\Lambda)| = \frac{1}{m_1! \cdots m_q!} \binom{n}{n_1 \cdots n_m} \prod_{j=1}^{m} d_{\lambda^{(j)}} \prod_{j=1}^{q} d_{\mu^{(j)}}.$$

PROOF. This follows by combining (87) and (89) from the proof of the Lemma. □

We now start out the construction of the cellular basis elements of $\mathcal{E}_n(q)$. Recall that with $\Lambda = (\boldsymbol{\lambda} \mid \mathbf{u}) \in \mathcal{L}_n$ we have associated the set of multiplicities $\{m_i\}$ of equal $\lambda^{(i)}$'s. We also associate with Λ the set of multiplicities $\{k_i\}$ of equal block sizes $|\lambda^{(i)}|$, as in the Corollary. That is, k_1 is the maximal i such that $|\lambda^{(1)}| = |\lambda^{(2)}| = \ldots = |\lambda^{(i)}|$, whereas k_2 is the maximal i such that $|\lambda^{(k_1+1)}| = |\lambda^{(k_1+2)}| = \ldots = |\lambda^{(k_1+i)}|$.

Let $\mathfrak{S}_\Lambda \leq \mathfrak{S}_n$ be the stabilizer subgroup of the set partition $A_{\boldsymbol{\lambda}}$ that was introduced in (53). Then the two sets of multiplicities give rise to subgroups $\mathfrak{S}_\Lambda^k$ and $\mathfrak{S}_\Lambda^m$ of $\mathfrak{S}_\Lambda$ where $\mathfrak{S}_\Lambda^k$ consists of the order preserving permutations of the equally sized blocks of $A_{\boldsymbol{\lambda}}$, whereas $\mathfrak{S}_\Lambda^m$ consists of the order preserving permutations of those blocks of $A_{\boldsymbol{\lambda}}$ that correspond to equal $\lambda^{(i)}$'s. Clearly we have $\mathfrak{S}_\Lambda^m \leq \mathfrak{S}_\Lambda^k \leq \mathfrak{S}_\Lambda$.

We next show that the group algebras $S\mathfrak{S}_\Lambda^m$ and $S\mathfrak{S}_\Lambda^k$ can be viewed as subalgebras of $\mathcal{E}_n(q)$. Suppose that σ_i is the simple transposition of $\mathfrak{S}_\Lambda^k$ that interchanges the consecutive blocks I_i and I_{i+1} of $A_{\boldsymbol{\lambda}}$. We have that $|I_i| = |I_{i+1}|$ and so we can write

$$I_i = \{c+1, c+2, \dots, c+a\} \text{ and } I_{i+1} = \{c+a+1, c+a+2, \dots, c+2a\}$$

for some c where $a := |I_1|$. We now define

$$B_i := (c+1, c+a+1)(c+2, c+a+2) \cdots (c+a, c+2a). \tag{90}$$

Then B_i interchanges the numbers of I_i and I_{i+1} pairwise, that is $(c+1)B_i = c+a+1$, and so on. For $i > j$ we set $s_{i,j} := s_{c+i} s_{c+i-1} \ldots s_{c+j}$ and can then write B_i in terms of simple transpositions as follows

$$B_i = s_{a,1} s_{a+1,2} \ldots s_{2a-1,a}. \tag{91}$$

Inspired by this formula we define $\mathbb{B}_i \in \mathcal{E}_n(q)$ as follows

$$\mathbb{B}_i := \mathbb{E}_{A_\Lambda} g_{a,1} g_{a+1,2} \dots g_{2a-1,a} \tag{92}$$

where $g_{i,j} := g_{i+c} g_{i-1+c} \dots g_{j+c}$. We can now state our next result.

Lemma 45 Suppose that $\Lambda \in \mathcal{L}_n(\alpha)$. Then the rule $s_i \mapsto \mathbb{B}_i$ induces algebra embeddings

$$\iota : S\mathfrak{S}_\Lambda^k \to \mathcal{E}_n^\alpha(q), \quad \iota : S\mathfrak{S}_\Lambda^m \to \mathcal{E}_n^\alpha(q).$$

PROOF. It is easy to see that $\mathbb{B}_i$ belongs to $\mathcal{E}_n^\alpha(q)$: for example we get via (2) of Proposition 38 that the factor $\mathbb{E}_{A_\Lambda}$ of $\mathbb{B}_i$ can be moved to the extreme right of (92), and the claim follows.

Now the B_i's satisfy the symmetric group relations (1), (2) and (3) as one easily checks from (90). But this fact can also be deduced from (91) using only the symmetric group relations on the s_i's. Since $\mathbb{B}_i$ is obtained from B_i by replacing s_i with g_i we obtain a proof of the symmetric group relations for $\mathbb{B}_i$'s, once we can show that the g_i's involved in showing the symmetric group relations for the B_i's satisfy the quadratic relation (3): after all the braid relations on the g_i's are already satisfied.

Fix an f and let $\mathbb{B}_i = \mathbb{E}_{A_\Lambda} g_{i_1} \dots g_{i_{f-1}} g_{i_f} \dots g_{i_p}$ be the expansion of $\mathbb{B}_i$ in terms of the g_i's. Using Proposition 38 once again, this can also be written in the form $\mathbb{B}_i = g_{i_1} \dots g_{i_{f-1}} \mathbb{E}_{A_\Lambda s_{i_1} \dots s_{i_{f-1}}} g_{i_f} \dots g_{i_p}$ and so $\mathbb{E}_{A_\Lambda s_{i_1} \dots s_{i_{f-1}}} g_{i_f}^2 = \mathbb{E}_{A_\Lambda s_{i_1} \dots s_{i_{f-1}}}$ because i_f and i_{f+1} occur in different blocks of $A_\Lambda s_{i_1} \dots s_{i_{f-1}}$.

On the other hand, the proof of $B_i^2 = 1$ using the symmetric group relations on the s_i's only involves quadratic relations of the indicated type, as one easily sees from the definition of B_i, and so it also gives a proof of $\mathbb{B}_i^2 = 1$. Similarly the other relations (1) and (2) are obtained. □

Suppose that $y \in \mathfrak{S}_\Lambda^k$ and let $y := s_{i_1} \dots s_{i_k}$ be a reduced expression. Then we define $B_y := B_{i_1} \dots B_{i_k}$ and $\mathbb{B}_y := \mathbb{B}_{i_1} \dots \mathbb{B}_{i_k}$. Note that, by the above Lemma, $\mathbb{B}_y \in \mathcal{E}_n(q)$ is independent of the chosen reduced expression.

Recall that for any algebra $\mathcal{A}$, the wreath product $\mathcal{A} \wr \mathfrak{S}_m$ is defined as the semidirect product $\mathcal{A}^{\otimes m} \rtimes \mathfrak{S}_m$ where $\mathfrak{S}_m$ acts on $\mathcal{A}^{\otimes m}$ via place permutation. Let still $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n$ and let $\|\boldsymbol{\lambda}\| = (a_1^{k_1}, \dots, a_r^{k_r})$ with strictly increasing a_i's. With this notation we have that

$$S\mathfrak{S}_\Lambda = S\mathfrak{S}_{a_1} \wr \mathfrak{S}_{k_1} \otimes \dots \otimes S\mathfrak{S}_{a_r} \wr \mathfrak{S}_{k_r}. \tag{93}$$

The inclusion of $\mathfrak{S}_\Lambda^k = \mathfrak{S}_{k_1} \times \dots \times \mathfrak{S}_{k_r}$ in $\mathfrak{S}_\Lambda$ is induced by the map that takes $\mathfrak{S}_{k_i}$ to the second factor of $S\mathfrak{S}_{a_i} \wr \mathfrak{S}_{k_i}$. We can now extend the last Lemma as follows.

Lemma 46 Let $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n$ as above with $\|\boldsymbol{\lambda}\| = (a_1^{k_1}, \dots, a_r^{k_r})$ and set $\mathcal{H}_\alpha^{wr}(q) := \mathcal{H}_{a_1}(q) \wr \mathfrak{S}_{k_1} \otimes \dots \otimes \mathcal{H}_{a_r}(q) \wr \mathfrak{S}_{k_r}$. Then $\mathcal{H}_\alpha^{wr}(q)$ is a subalgebra of $\mathcal{E}_n^\alpha(q)$. In the $\mathcal{H}_{a_i}(q)$-factors the inclusion is given by $g_i \mapsto g_i$, whereas in the $\mathfrak{S}_{k_i}$-factors the inclusion is given by ι from the previous Lemma.

Let us now start the construction of the cellular basis for $\mathcal{E}_n(q)$. As in the Yokonuma-Hecke algebra case, we first construct, for each $\Lambda \in \mathcal{L}_n$, an element m_Λ that acts as the starting point of the basis. Suppose that $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$ is as above with $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(q)})$. We then define m_Λ as follows

$$m_\Lambda := \mathbb{E}_{A_\Lambda} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}}. \tag{94}$$

Let us explain the factors of the product. Firstly, $\mathbb{E}_{A_\Lambda}$ is the idempotent defined in (80). Secondly, $x_{\boldsymbol{\lambda}}$ is the Murphy element already considered in the section on the

Yokonuma-Hecke algebra, that is $x_{\boldsymbol{\lambda}} := \sum_{w\in\mathfrak{S}_{\boldsymbol{\lambda}}} q^{\ell(w)} g_w$, where $\mathfrak{S}_{\boldsymbol{\lambda}}$ is the row stabilizer of the multitableau $\mathbf{t}^{\boldsymbol{\lambda}}$. Finally, in order to explain the factor $b_{\boldsymbol{\mu}}$ we use that $\mathfrak{S}_\Lambda^m \cong \mathfrak{S}_{k_1} \times \ldots \times \mathfrak{S}_{k_q}$ where $k_i = |\mu^{(i)}|$ is the order of the partition $\mu^{(i)}$. Let $x_{\mu^{(i)}} \in \mathfrak{S}_{k_i}$ be the $q = 1$ specialization of the Murphy element corresponding to $\mu^{(i)}$. We view $x_{\mu^{(j)}}$ as an element of $\mathfrak{S}_\Lambda^m$ via the above isomorphism and then define $b_{\boldsymbol{\mu}} := \prod_{i=1}^q b_{\boldsymbol{\mu}^{(i)}}$ where $b_{\boldsymbol{\mu}^{(i)}} := \iota(x_{\mu^{(i)}})$ for $\iota : S\mathfrak{S}_\Lambda^m \to \mathcal{E}_n(q)$ the embedding of the Lemma.

Let $\mathbbm{t}^\Lambda$ be the Λ-tableau given in the obvious way as $\mathbbm{t}^\Lambda := (\mathbf{t}^{\boldsymbol{\lambda}} \mid (t^{\mu^{(1)}}, \ldots, t^{\mu^{(q)}}))$. Then $\mathbbm{t}^\Lambda$ is a maximal Λ-tableau, that is the only standard Λ-tableau $\mathbbm{t}$ satisfying $\mathbbm{t} \trianglerighteq \mathbbm{t}^\Lambda$ is $\mathbbm{t}^\Lambda$ itself. For $\mathbbm{s} = (\mathfrak{s} \mid (u_1, \ldots, u_q))$ a Λ-tableau we define $d(\mathbbm{s}) := (d(\mathbbm{s}) \mid (d(u_1), \ldots, d(u_q))$ where $d(\mathfrak{s}) \in \mathfrak{S}_n$ is the element that maps $\mathbf{t}^{\boldsymbol{\lambda}}$ to $\mathfrak{s}$ and $d(u_i) \in \mathfrak{S}_{k_i}$ the element that maps $t^{\mu^{(i)}}$ to u_i. Suppose now that $\mathbf{u} = (u_1, \ldots, u_q)$ is the second component of the row standard Λ-tableaux $\mathbbm{s}$; then we set

$$\mathbb{B}_{d(\mathbf{u})} := \mathbb{B}_{d(u_1)} \cdots \mathbb{B}_{d(u_q)}.$$

Finally, we define the main object of this section. For $\mathbbm{s} = (\mathfrak{s} \mid \mathbf{u})$, $\mathbbm{t} = (\mathbf{t} \mid \mathbf{v})$ row standard Λ-tableaux we define

$$m_{\mathbbm{s}\mathbbm{t}} := g^*_{d(\mathfrak{s})} \mathbb{E}_{A_\Lambda} \mathbb{B}^*_{d(\mathbf{u})} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}} \mathbb{B}_{d(\mathbf{v})} g_{d(\mathbf{t})}. \tag{95}$$

Our aim is to prove that the $m_{\mathbbm{s}\mathbbm{t}}$'s, with $\mathbbm{s}$ and $\mathbbm{t}$ running over standard Λ-tableaux, form a cellular basis for $\mathcal{E}_n(q)$. To achieve this goal we first need to work out commutation rules between the various ingredients of $m_{\mathbbm{s}\mathbbm{t}}$. The rules shall be formulated in terms of a certain $\circ$-action on tableaux that we explain now.

From now on, when confusion should not be possible, we shall write $\mathfrak{s}y$ *for* $\mathfrak{s}B_y$, *where* $\mathfrak{s}$ *is the first part of a* Λ*-tableau and* $y \in \mathfrak{S}_\Lambda^k$.

Let $\mathbbm{s} = (\mathfrak{s} \mid \mathbf{u})$ be a Λ-tableau. We then define a new multitableau $y \circ \mathfrak{s}$ as follows. Set first $\mathfrak{s}_1 := \mathfrak{s}y^{-1} = (s_1^{(1)}, \ldots, s_1^{(m)})$. Then $y \circ \mathfrak{s}$ is given by the formula.

$$y \circ \mathfrak{s} := (s_1^{(1)y}, \ldots, s_1^{(m)y}). \tag{96}$$

With this notation we have the following Lemma which is easy to verify.

Lemma 47 The map $(y, \mathfrak{s}) \mapsto y \circ \mathfrak{s}$ defines a left action of $\mathfrak{S}_\Lambda^k$ on the set of multitableaux $\mathfrak{s}$ such that $Shape(\mathfrak{s}) =_1 \boldsymbol{\lambda}$ where $\boldsymbol{\lambda}$ is the first part if Λ; that is $Shape(\mathfrak{s})$ and $\boldsymbol{\lambda}$ are equal multipartitions up to a permutation. Moreover, if $\mathfrak{s}$ is of the initial kind then also $y \circ \mathfrak{s}$ is of the initial kind, and if $y \in \mathfrak{S}_\Lambda^m$ then $y \circ \mathfrak{s} = \mathfrak{s}$.

Example 48 We give an example to illustrate the action. As can be seen, it permutes the partitions of the multitableau, but keeps the numbers. Consider

$$\mathfrak{s} := \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array}, \begin{array}{|c|}\hline 5 \\ \hline 6 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 7 & 9 \\ \hline 8 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|}\hline 10 & 11 & 12 \\ \hline \end{array}, \begin{array}{|c|}\hline 13 \\ \hline 14 \\ \hline 15 \\ \hline \end{array} \right) \quad \text{and} \quad y := s_1 s_2 s_1 s_4 s_5.$$

We first note that $\mathfrak{s}y^{-1} = \left(\begin{array}{|c|c|}\hline 5 & 6 \\ \hline \end{array}, \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 10 & 12 \\ \hline 11 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|}\hline 13 & 14 & 15 \\ \hline \end{array}, \begin{array}{|c|}\hline 7 \\ \hline 8 \\ \hline 9 \\ \hline \end{array} \right)$. Then we have

$$y \circ \mathfrak{s} = \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 5 & 6 \\ \hline \end{array}, \begin{array}{|c|}\hline 7 \\ \hline 8 \\ \hline 9 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 10 & 12 \\ \hline 11 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|}\hline 13 & 14 & 15 \\ \hline \end{array} \right)$$

The next Lemma gives the promised commutation formulas.

Lemma 49 Suppose $\mathbb{s} = (\mathfrak{s} \,|\, \mathbf{u})$ and $\mathbb{t} = (\mathbf{t} \,|\, \mathbf{v})$ are Λ-tableaux such that $\mathfrak{s}$ and $\mathbf{t}$ are of the initial type and suppose that $y \in \mathfrak{S}_\Lambda^k$. Then we have the following formulas in $\mathcal{E}_n(q)$

(1) $\mathbb{E}_{A_\Lambda}\mathbb{B}_y g_{d(\mathfrak{s})} = \mathbb{E}_{A_\Lambda} g_{d(y\circ\mathfrak{s})}\mathbb{B}_y$

(2) $\mathbb{E}_{A_\Lambda}\mathbb{B}_y x_{\mathfrak{s}\mathbf{t}} = \mathbb{E}_{A_\Lambda} x_{y\circ\mathfrak{s}, y\circ\mathbf{t}}\mathbb{B}_y$.

PROOF. Suppose first that σ_i is a simple transposition of $\mathfrak{S}_\Lambda^k$ as above in (90), (91) and (92), with $c=0$. In other words, σ_i interchanges the blocks I_i and I_{i+1} of A_Λ. We then first prove that for $i \neq a, 2a$ we have that

$$g_{a,1}g_{a+1,2}\cdots g_{2a-1,a}g_i = g_{i\sigma_i}g_{a,1}g_{a+1,2}\cdots g_{2a-1,a} \tag{97}$$

where we write σ_i for iB_i. Using the braid relations one first checks that $g_{a,b}g_i = g_{i-1}g_{a,b}$ for $i \in \{b+1,\ldots,a\}$. Hence, we get that (97) holds if $i \in I_2$. On the other hand, using the braid relations once again one gets that

$$(g_{a,1}g_{a+1,2}\cdots g_{2a-1,a})^* = g_{a,1}g_{a+1,2}\cdots g_{2a-1,a}$$

(actually the commuting braid relations are sufficient for this) and hence the $i \in I_1$ case of (97) follows by applying $*$ to the $i \in I_2$ case. The remaining cases of (97), where $i \notin I_1 \cup I_2$, are easy.

But combining (97) and (92) we get the formula

$$\mathbb{E}_{A_\Lambda}\mathbb{B}_y g_k = \mathbb{E}_{A_\Lambda} g_{kB_y^{-1}}\mathbb{B}_y \tag{98}$$

valid for the generators g_k of $\mathfrak{S}_\Lambda^k$. With this we can prove prove (1) of the Lemma. Indeed, writing $\mathfrak{s} = (s^{(1)},\ldots,s^{(m)})$ we get, by using that $\mathfrak{s}$ is of the initial kind, that $g_{d(\mathfrak{s})} = g_{d(s^{(1)})}\cdots g_{d(s^{(m)})}$, and so we have via (98) that

$$\mathbb{E}_{A_\Lambda}\mathbb{B}_y g_{d(\mathfrak{s})} = \mathbb{E}_{A_\Lambda}\mathbb{B}_y g_{d(s^{(1)})}\cdots g_{d(s^{(m)})} = \mathbb{E}_{A_\Lambda} g_{d(s^{(1)}y^{-1})}\cdots g_{d(s^{(m)}y^{-1})}\mathbb{B}_y.$$

But clearly the elements $g_{d(s_i y^{-1})}$ commute and so we get that

$$g_{d(y\circ\mathfrak{s})} = g_{d(s^{(1)}y^{-1})}\cdots g_{d(s^{(m)}y^{-1})}$$

and (1) follows. On the other hand, applying $*$ to (1) and using that $\mathbb{B}_y^* = \mathbb{B}_{y^{-1}}$, we find that $\mathbb{E}_{A_\Lambda}\mathbb{B}_{y^{-1}}g^*_{d(y\circ\mathfrak{s})} = \mathbb{E}_{A_\Lambda} g^*_{d(\mathfrak{s})}\mathbb{B}_{y^{-1}}$, or by change of variable

$$\mathbb{E}_{A_\Lambda}\mathbb{B}_y g^*_{d(\mathfrak{s})} = \mathbb{E}_{A_\Lambda} g^*_{d(y\circ\mathfrak{s})}\mathbb{B}_y.$$

Moreover, by using formula (98) and the commutativity of the factors of $x_{\boldsymbol{\lambda}}$ it is easy to check that $\mathbb{E}_{A_\Lambda}\mathbb{B}_y x_{\boldsymbol{\lambda}} = \mathbb{E}_{A_\Lambda} x_{\boldsymbol{\mu}}\mathbb{B}_y$, where $\boldsymbol{\mu} = Shape(y \circ \mathbf{t}^{\boldsymbol{\lambda}})$, and so also (2) follows. $\square$

As an immediate consequence of the Lemma we get that the factor $x_{\boldsymbol{\lambda}}$ of m_Λ commutes with the factors $\mathbb{B}^*_{d(\mathbf{u})}$, $b_{\boldsymbol{\mu}}$ and $\mathbb{B}_{d(\mathbf{v})}$. Further, by Proposition 38(1), we have that $\mathbb{E}_{A_\Lambda} g_w = g_w \mathbb{E}_{A_\Lambda}$ for all $w \in \mathfrak{S}_\Lambda^k$. Hence, we obtain that

$$m^*_{\mathbb{s}\mathbb{t}} = g^*_{d(\mathbf{t})}\mathbb{B}^*_{d(\mathbf{v})} b^*_{\boldsymbol{\mu}} x^*_{\boldsymbol{\lambda}}\mathbb{B}_{d(\mathbf{u})}\mathbb{E}^*_{A_\Lambda} g_{d(\mathfrak{s})} = g^*_{d(\mathbf{t})}\mathbb{E}_{A_\Lambda}\mathbb{B}^*_{d(\mathbf{v})} x_{\boldsymbol{\lambda}} b_{\boldsymbol{\mu}}\mathbb{B}_{d(\mathbf{u})} g_{d(\mathfrak{s})} = m_{\mathbb{t}\mathbb{s}}. \tag{99}$$

The following Lemma is the $\mathcal{E}_n(q)$-version of Lemma 26 in the Yokonuma-Hecke algebra case.

Lemma 50 Suppose that $\Lambda \in \mathcal{L}_n$ and that $\mathbb{s}$ and $\mathbb{t}$ are row standard Λ-tableaux. Then for every $h \in \mathcal{E}_n(q)$ we have that $m_{\mathbb{s}\mathbb{t}}h$ is a linear combination of terms of the form $m_{\mathbb{s}\mathbb{v}}$ where $\mathbb{v}$ is a row standard Λ-tableau. A similar statement holds for $hm_{\mathbb{s}\mathbb{t}}$.

PROOF. The idea is to repeat the arguments of Lemma 26. It is enough to consider the $m_{\mathbb{s}\mathbb{t}}h$ case. Using the remarks prior to the Lemma we have that

$$m_{\mathbb{s}\mathbb{t}} = g^*_{d(\mathfrak{s})}\mathbb{E}_{A_\Lambda}\mathbb{B}^*_{d(\mathbf{u})}x_{\boldsymbol{\lambda}}b_{\boldsymbol{\mu}}\mathbb{B}_{d(\mathbf{v})}\,g_{d(\mathfrak{t})} = g^*_{d(\mathfrak{s})}\mathbb{B}^*_{d(\mathbf{u})}b_{\boldsymbol{\mu}}\mathbb{B}_{d(\mathbf{v})}\mathbb{E}_{A_\Lambda}x_{\boldsymbol{\lambda}}\,g_{d(\mathfrak{t})}. \tag{100}$$

We may assume that $h = E_A g_w$ since these elements form a basis for $\mathcal{E}_n(q)$. On the other hand, by Proposition 38 we have that right multiplication with E_A maps $m_{\mathbb{s}\mathbb{t}}$ to either $m_{\mathbb{s}\mathbb{t}}$ itself or to 0 and so we may actually assume that $h = g_w$. But repeating the argument from Lemma 26 we now get that $m_{\mathbb{s}\mathbb{t}}h$ can be written as a linear combination of terms

$$g^*_{d(\mathfrak{s})}\mathbb{B}^*_{d(\mathbf{u})}b_{\boldsymbol{\mu}}\mathbb{B}_{d(\mathbf{v})}\mathbb{E}_{A_\Lambda}x_{\boldsymbol{\lambda}}\,g_{d(\mathfrak{t}_1)}$$

where $\mathfrak{t}_1$ is a row standard $\boldsymbol{\lambda}$-tableau. Commuting $x_{\boldsymbol{\lambda}}$ back as in (100) we get that $m_{\mathbb{s}\mathbb{t}}h$ is a linear combination of $m_{\mathbb{s}\mathbb{t}_1}$'s where the $\mathbb{t}_1$'s are row standard Λ-tableaux.

□

Our next Lemma is the analogue for $\mathcal{E}_n(q)$ of Lemma 27. It is the key Lemma for our results on $\mathcal{E}_n(q)$.

Lemma 51 Suppose that $\Lambda \in \mathcal{L}_n$ and that $\mathbb{s}$ and $\mathbb{t}$ are row standard Λ-tableaux. Then there are standard tableaux $\mathbb{u}$ and $\mathbb{v}$ such that $\mathbb{u} \trianglerighteq \mathbb{s}, \mathbb{v} \trianglerighteq \mathbb{t}$ and such that $m_{\mathbb{s}\mathbb{t}}$ is a linear combination of the elements $m_{\mathbb{u}\mathbb{v}}$.

PROOF. Let $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu})$, $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$ and $\mathbb{t} = (\mathfrak{t} \mid \mathbf{v})$. Then we have

$$m_{\mathbb{s}\mathbb{t}} = g^*_{d(\mathfrak{s})}\mathbb{E}_{A_\Lambda}\mathbb{B}^*_{d(\mathbf{u})}b_{\boldsymbol{\mu}}x_{\boldsymbol{\lambda}}\mathbb{B}_{d(\mathbf{v})}g_{d(\mathfrak{t})}. \tag{101}$$

Suppose first that standardness fails for $\mathbb{s}$ or $\mathbb{t}$. The basic idea is then to proceed as in the proof of Lemma 27. There exist multitableaux $\mathfrak{s}_0$ and $\mathfrak{t}_0$ of the initial kind together with $w_{\mathfrak{s}}, w_{\mathfrak{t}} \in \mathfrak{S}_n$ such that $d(\mathfrak{s}) = d(\mathfrak{s}_0)w_{\mathfrak{s}}$, $d(\mathfrak{t}) = d(\mathfrak{t}_0)w_{\mathfrak{t}}$ and $\ell(d(\mathfrak{s})) = \ell(d(\mathfrak{s}_0)) + \ell(w_{\mathfrak{s}})$ and $\ell(\mathfrak{t})) = \ell(d(\mathfrak{t}_0)) + \ell(w_{\mathfrak{t}})$. That is, $w_{\mathfrak{s}}$ and $w_{\mathfrak{t}}$ are distinguished right coset representatives for $\mathfrak{S}_{\|\boldsymbol{\lambda}\|}$ in $\mathfrak{S}_n$ and (101) becomes

$$m_{\mathbb{s}\mathbb{t}} = g^*_{w_{\mathfrak{s}}}g^*_{d(\mathfrak{s}_0)}\mathbb{E}_{A_\Lambda}\mathbb{B}^*_{d(\mathbf{u})}x_{\boldsymbol{\lambda}}b_{\boldsymbol{\mu}}\mathbb{B}_{d(\mathbf{v})}g_{d(\mathfrak{t}_0)}g_{w_{\mathfrak{t}}} \tag{102}$$

since the two middle terms commute. Note that the factor $\mathbb{E}_{A_\Lambda}$ commutes with all other except the two extremal factors of (102). Expanding $b_{\boldsymbol{\mu}}\mathbb{B}_{d(\mathbf{v})}$ completely as a linear combination of $\mathbb{B}_{y_{\mathfrak{s}}}$'s where $y_{\mathfrak{s}} \in \mathfrak{S}^m_\Lambda$ and writing $\mathbb{B}_{y_{\mathfrak{t}}} := \mathbb{B}_{d(\mathbf{u})}$ where also $y_{\mathfrak{t}} \in \mathfrak{S}^m_\Lambda$ we get via Proposition 49 that (102) becomes a linear combination of terms

$$g^*_{w_{\mathfrak{s}}}\mathbb{B}^*_{y_{\mathfrak{s}}}x_{y_{\mathfrak{s}}\circ\mathfrak{s}_0, y_{\mathfrak{t}}\circ\mathfrak{t}_0}\mathbb{B}_{y_{\mathfrak{t}}}g_{w_{\mathfrak{t}}} \tag{103}$$

where $y_{\mathfrak{s}} \circ \mathfrak{s}_0$ and $y_{\mathfrak{t}} \circ \mathfrak{t}_0$, by Proposition 47, are of the initial type. For each appearing $y_{\mathfrak{s}}$ we have that $B_{y_{\mathfrak{s}}}$ is a right coset representative for $\mathfrak{S}_{\|\boldsymbol{\lambda}\|}$ and moreover, although in general $\ell(B_{y_{\mathfrak{s}}}w_{\mathfrak{s}}) \neq \ell(B_{y_{\mathfrak{s}}}) + \ell(w_{\mathfrak{s}})$ we have that

$$\mathbb{E}_{A_\Lambda}g_{B_{y_{\mathfrak{s}}}}g_{w_{\mathfrak{s}}} = \mathbb{E}_{A_\Lambda}g_{B_{y_{\mathfrak{s}}}w_{\mathfrak{s}}}$$

and a similar statement is true for each appearing $y_{\mathfrak{t}}$. This is so because the action of $g_{B_{y_{\mathfrak{s}}}}$ and g_{w_s}, when written out as a product of simple transpositions, always involves different blocks of A_Λ corresponding to the first two cases of Lemma 23, the symmetric group cases. Thus (103) beomes a linear combination of terms

$$g^*_{y_{\mathfrak{s},1}}\mathbb{E}_{A_\Lambda}x_{y_{\mathfrak{s}}\circ\mathfrak{s}_0, y_{\mathfrak{t}}\circ\mathfrak{t}_0}g_{y_{\mathfrak{t},1}} \tag{104}$$

where $y_{\mathfrak{s},1} := y_{\mathfrak{s}} w_{\mathfrak{s}}$ and $y_{\mathfrak{t},1} := y_{\mathfrak{t}} w_{\mathfrak{t}}$ are distinguished right coset representatives for $\mathfrak{S}_{\|\boldsymbol{\lambda}\|}$. Just as in the Yokonuma-Hecke algebra case, we now apply Murphy's Theorem [34, Theorem 4.18] on $x_{y_{\mathfrak{s}}\circ\mathfrak{s}_0, y_{\mathfrak{t}}\circ\mathfrak{t}_0}$, thus rewriting it as a linear combination of $x_{\mathfrak{s}_1\mathfrak{t}_1}$ where $\mathfrak{s}_1$ and $\mathfrak{t}_1$ are standard $\boldsymbol{\nu}$-multitableaux of the initial kind, where $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(m)})$ say, such that $\mathfrak{s}_1 \trianglerighteq y_{\mathfrak{s}} \circ \mathfrak{s}_0$ and $\mathfrak{t}_1 \trianglerighteq y_{\mathfrak{t}} \circ \mathfrak{t}_0$. We then get that (103) is a linear combination of such terms

$$g^*_{y_{\mathfrak{s},1}} \mathbb{E}_{A_\Lambda} x_{\mathfrak{s}_1,\mathfrak{t}_1} g_{y_{\mathfrak{t},1}}. \tag{105}$$

Here $\boldsymbol{\nu}$ need not be an increasing multipartition and our task is to fix this problem.

We determine a $\sigma \in \mathfrak{S}^k_\Lambda$ such that the multipartition $\boldsymbol{\nu}^{ord} := (\nu^{(1)\sigma}, \dots, \nu^{(m)\sigma})$ is increasing. Then, using (2) of Lemma 49 we get that (105) is equal to

$$g^*_{y_{\mathfrak{s},1}} \mathbb{B}^*_\sigma \mathbb{B}_\sigma x_{\mathfrak{s}_1,\mathfrak{t}_1} g_{y_{\mathfrak{t},1}} = g^*_{y_{\mathfrak{s},1}} \mathbb{B}^*_\sigma x_{\sigma\circ\mathfrak{s}_1,\sigma\circ\mathfrak{t}_1} \mathbb{B}_\sigma g_{y_{\mathfrak{t},1}} = g^*_{y_{\mathfrak{s},2}} x_{\sigma\circ\mathfrak{s}_1,\sigma\circ\mathfrak{t}_1} g_{y_{\mathfrak{t},2}} \tag{106}$$

where $y_{\mathfrak{s},2} := B_\sigma y_{\mathfrak{s},1}$ and $y_{\mathfrak{t},2} := B_\sigma y_{\mathfrak{t},1}$. Here $\mathfrak{t}^{\boldsymbol{\nu}^{ord}} y_{\mathfrak{s},2}$ and $\mathfrak{t}^{\boldsymbol{\nu}^{ord}} y_{\mathfrak{t},2}$ are standard $\boldsymbol{\nu}^{ord}$-multitableaux, but they need not be increasing. But letting $\mathfrak{S}^{m'}$ be the subgroup of $\mathfrak{S}^k_\Lambda$ that permutes equal $\nu^{(i)}$'s we can find $\sigma_1, \sigma_2 \in \mathfrak{S}^{m'}_\Lambda$ such that $\mathfrak{t}^{\boldsymbol{\nu}^{ord}} B_{\sigma_1} y_{\mathfrak{s},2}$ and $\mathfrak{t}^{\boldsymbol{\nu}^{ord}} B_{\sigma_2} y_{\mathfrak{t},2}$ are increasing $\boldsymbol{\nu}^{ord}$-tableaux. With these choices, (106) becomes

$$g^*_{y_{\mathfrak{s},2}} x_{\sigma\circ\mathfrak{s}_1,\sigma\circ\mathfrak{t}_1} \mathbb{B}^*_{\sigma_1} \mathbb{B}_{\sigma_2} g_{y_{\mathfrak{t},2}} \tag{107}$$

where $y_{\mathfrak{s},3} := B_{\sigma_1} y_{\mathfrak{s},2}$ and $y_{\mathfrak{s},3} := B_{\sigma_2} y_{\mathfrak{t},2}$, and where we used (2) of Lemma 49 to show that $\sigma \circ \mathfrak{s}_1$ and $\sigma \circ \mathfrak{t}_1$ are unchanged by the commutation with $\mathbb{B}^*_{\sigma_1} \mathbb{B}_{\sigma_2}$. We now set $\mathfrak{s}_3 := \mathfrak{t}^{\boldsymbol{\nu}^{ord}} d(\sigma \circ \mathfrak{s}_1) y_{\mathfrak{s},3}$, where obviously $d(\sigma \circ \mathfrak{s}_1)$ is calculated with respect to $\mathfrak{t}^{\boldsymbol{\nu}^{ord}}$, and similarly $\mathfrak{t}_3 := \mathfrak{t}^{\boldsymbol{\nu}^{ord}} d(\sigma \circ \mathfrak{t}_1) y_{\mathfrak{t},3}$: here we use once again Lemma 49 to see that $d(\sigma \circ \mathfrak{t}_1)$ and $\mathbb{B}^*_{\sigma_1} \mathbb{B}_{\sigma_2}$ commute. Then $\mathfrak{s}_3$ and $\mathfrak{t}_3$ are increasing standard multitableaux of shape $\boldsymbol{\nu}^{ord}$ and we get that (107) is equal to

$$g^*_{d(\mathfrak{s}_3)} x_{\boldsymbol{\nu}^{ord}} \mathbb{B}^*_{\sigma_1} \mathbb{B}_{\sigma_2} g_{d(\mathfrak{t}_3)}. \tag{108}$$

In order to show that (101) has the form $m_{\mathbb{u}\mathbb{v}}$ stipulated by the Lemma, we must now treat the factors $\mathbb{B}^*_{\sigma_1} \mathbb{B}_{\sigma_2}$. But since $\mathfrak{S}^{m'}$ is a product of symmetric groups, $\mathbb{B}^*_{\sigma_1} \mathbb{B}_{\sigma_2}$ can simply be written as a linear combination of $x_{\mathbf{u}'\mathbf{v}'}$ of Murphy standard basis elements for that product, but of course without control over the involved partitions.

We must finally treat the case where standardness holds for $\mathfrak{s}$ and $\mathfrak{t}$, but fails for $\mathbf{u}$ or $\mathbf{v}$. But this case is much easier, since we can here apply Murphy's theory directly, thus expanding the nonstandard terms in terms of standard terms. Finally, the order condition of the Lemma follows directly from the definitions. □

We are now ready to state and prove the main Theorem of this section.

Theorem 52 Let $\mathcal{BT}_n := \{m_{\mathbb{s}\mathbb{t}} \mid \mathbb{s}, \mathbb{t} \in \mathrm{Std}(\Lambda), \Lambda \in \mathcal{L}_n\}$. Then $(\mathcal{BT}_n, \mathcal{L}_n)$ is a cellular basis for $\mathcal{E}_n(q)$.

Proof. Using Lemma 2, that as already mentioned is true for $\mathcal{E}_n(q)$ too, together with the $\{E_A g_w\}$-basis for $\mathcal{E}_n(q)$, we find that the set $\{g_w E_{A_\Lambda} g_{w^1} \mid A \in \mathcal{SP}_n, w, w^1 \in \mathfrak{S}_n\}$ generates $\mathcal{E}_n(q)$ over S. Thus, letting $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n$ vary over pairs of one column partitions and $\mathbb{s}, \mathbb{t}$ over row standard Λ-tableaux, we get that the corresponding $m_{\mathbb{s}\mathbb{t}}$ generate $\mathcal{E}_n(q)$ over S. But then, using the last two Lemmas, we deduce that the elements from $\mathcal{BT}_n$ generate $\mathcal{E}_n(q)$ over S. On the other hand, by Lemma 43 these elements have cardinality equal to $\dim \mathcal{E}_n(q)$, and so they indeed form a basis for $\mathcal{E}_n(q)$, as can be seen by repeating the argument of Theorem 32.

The $*$-condition for cellularity has already been checked above in (99). Finally, to show the multiplication condition for $\mathcal{BT}_n$ to be cellular, we can repeat the argument from the Yokonuma-Hecke algebra case. Indeed, to $\Lambda = (\boldsymbol{\lambda} \mid \boldsymbol{\mu}) \in \mathcal{L}_n$ we have associated the Λ-tableau $\mathbb{t}^{\Lambda}$ and have noticed that the only standard Λ-tableau $\mathbb{t}$ satisfying $\mathbb{t} \trianglerighteq \mathbb{t}^{\Lambda}$ is $\mathbb{t}^{\Lambda}$ itself. The Theorem follows from this just like in the Yokonuma-Hecke algebra case. $\square$

Corollary 53 The dimension of the cell module $C(\Lambda)$ associated with $\Lambda \in \mathcal{L}_n$ is given by the formula of Corollary 44.

Corollary 54 Let α be a partition of n. Recall the set $\mathcal{L}_n(\alpha)$ introduced in the proof of Lemma 43. Then $\mathcal{BT}_n^{\alpha} := \{m_{\mathbb{s}\mathbb{t}} \mid \mathbb{s}, \mathbb{t} \in \mathrm{Std}(\Lambda), \Lambda \in \mathcal{L}_n(\alpha)\}$ is a cellular basis for $\mathcal{E}_n^{\alpha}(q)$.

The next Corollary should be compared with the results of Geetha and Goodman, [16], who show that $\mathcal{A} \wr \mathfrak{S}_m$ is a cellular algebra, whenever $\mathcal{A}$ is a cyclic cellular algebra, meaning that the cell modules are all cyclic.

Definition 55 Let α be a partition of n and let $\Lambda \in \mathcal{L}_n(\alpha)$ and let $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$ be a Λ-tableau. Then we say that $\mathbb{s}$ is of wreath type for Λ if $\mathfrak{s} = \mathfrak{s}_0 B_y$ for some $y \in \mathfrak{S}_{\Lambda}^{k}$ and $\mathfrak{s}_0$ a multitableau of the initial kind. Moreover we define

$$\mathcal{BT}_n^{\alpha, wr} := \{m_{\mathbb{s}\mathbb{t}} \mid \mathbb{s}, \mathbb{t} \in \mathrm{Std}(\Lambda) \text{ of wreath type for } \Lambda \in \mathcal{L}_n(\alpha)\}.$$

Corollary 56 We have that $\mathcal{BT}_n^{\alpha, wr}$ is a cellular basis for the subalgebra $\mathcal{H}_{\alpha}^{wr}(q)$ of $\mathcal{E}_n^{\alpha}(q)$, introduced in Lemma 46.

PROOF. Clearly, we have that $\mathcal{BT}_n^{\alpha, wr} \subseteq \mathcal{H}_{\alpha}^{wr}(q)$. Moreover it follows for example from Geetha and Goodman's results in [16] that the cardinality of $\mathcal{BT}_n^{\alpha, wr}$ is equal to the dimension of $\mathcal{H}_{\alpha}^{wr}(q)$. On the other hand, one checks easily that Lemma 50 holds for $\mathcal{BT}_n^{\alpha, wr}$ with respect to $h \in \mathcal{H}_{\alpha}^{wr}(q)$ and that the straightening procedure of Lemma 51 applied to $m_{\mathbb{s}\mathbb{t}}$ for $\mathbb{s}, \mathbb{t}$ nonstandard tableaux of wreath type produces a linear combination of $m_{\mathbb{s}_0\mathbb{t}_0}$ where $\mathbb{s}, \mathbb{t}$ are standard tableaux and still of wreath type. Thus the proof of Theorem 52 also gives a proof of the Corollary. $\square$

Remark 57 Recall that we have fixed $\alpha = (a_1^{k_1}, \dots, a_r^{k_r})$ with strictly increasing a_i's such that $\mathfrak{S}_{\Lambda}^{k} = \mathfrak{S}_{k_1} \times \ldots \times \mathfrak{S}_{k_r}$. From Geetha and Goodman's cellular basis for $\mathcal{H}_{\alpha}^{wr}(q)$ we would have expected $\mathcal{BT}_n^{\alpha, wr}$ to be slightly different, namely given by pairs $(\mathfrak{s} \mid \mathbf{u})$ such that $\mathfrak{s}$ is a multitableaux of the initial kind whereas $\mathbf{u}$ is an r-tuple of multitableaux on the numbers $\{a_i k_i\}$. For example for $\Lambda = \Big(\big((1,1),(2),(2),(2,1)\big) \,\Big|\, \big((1),(1,1),(1)\big)\Big)$ we would have expected tableaux of the following form

$$\mathbb{t} := \left(\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 5 & 6 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 7 & 9 \\ \hline 8 \\ \cline{1-1} \end{array} \right) \,\middle|\, \left(\left(\begin{array}{|c|}\hline 2 \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array} \right), \left(\begin{array}{|c|}\hline 1 \\ \hline \end{array} \right) \right) \right) \tag{109}$$

where the shapes of the multitableaux occurring in $\mathbf{u}$ are given by the equally shaped tableaux of $\mathfrak{s}$. On the other hand, there is an obvious bijection between our standard tableaux of wreath type and the standard tableaux appearing in Geetha and Goodman's basis and so the cardinality of our basis is correct, which is enough for the above argument to work.

6.3. $\mathcal{E}_n(q)$ is a direct sum of matrix algebras

In this final subsection we use the cellular basis for $\mathcal{E}_n(q)$ to show that $\mathcal{E}_n(q)$ is isomorphic to a direct sum of matrix algebras in the spirit of Lusztig and Jacon-Poulain d'Andecy's result for the Yokonuma-Hecke algebra. We keep $\alpha = (a_1^{k_1}, \dots, a_r^{k_r})$ and $\mathfrak{S}_\Lambda^k = \mathfrak{S}_{k_1} \times \dots \times \mathfrak{S}_{k_r}$.

Suppose that $\Lambda \in \mathcal{L}_n(\alpha)$ and that $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$ is a standard Λ-tableau. Then we define $w_{\mathfrak{s}} \in \mathfrak{S}_n$ as the distinguished coset representative for $d(\mathfrak{s}) \in \mathfrak{S}_\alpha \backslash \mathfrak{S}_n$. We have that $\mathfrak{s}_0 w_{\mathfrak{s}} = \mathfrak{s}$ for $\mathfrak{s}_0$ of the initial kind and since $\mathfrak{s}_0$ and $w_{\mathfrak{s}}$ are unique we can define the Λ-tableau $\mathbb{s}_0 := (\mathfrak{s}_0 \mid \mathbf{u})$. Since $\mathfrak{s}$ is increasing we have for $y \in \mathfrak{S}_\Lambda^m$ that $B_y w_{\mathfrak{s}} \neq w_{\mathfrak{t}}$ for all $\mathbb{t} \in \mathcal{L}_n(\alpha)$ but we may have $B_y w_{\mathfrak{s}} = w_{\mathfrak{t}}$ for some $y \in \mathfrak{S}_\Lambda^k$. Let $\overline{w_{\mathfrak{s}}}$ be the orbit of $w_{\mathfrak{s}}$ under the action of $\mathfrak{S}_\Lambda^k$ that is $\overline{w_{\mathfrak{s}}} = \overline{w_{\mathfrak{t}}}$ if and only if $B_y w_{\mathfrak{s}} = w_{\mathfrak{t}}$ for some $y \in \mathfrak{S}_\Lambda^k$. We need the following Lemma.

Lemma 58 Suppose that $\Lambda, \overline{\Lambda} \in \mathcal{L}_n(\alpha)$ and that $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u})$ is a standard Λ-tableau and that $\mathbb{t} = (\mathfrak{t} \mid \mathbf{v})$ is a standard $\overline{\Lambda}$-tableau.

$$m_{\mathbb{t}^\Lambda \mathbb{s}} m_{\mathbb{t}\mathbb{t}^{\overline{\Lambda}}} = \begin{cases} \mathbb{E}_{A_\Lambda} x_{\mathbb{t}^\Lambda \mathbb{s}_0} x_{\mathbb{t}_0 \mathbb{t}^{\overline{\Lambda}}} & \text{if } \overline{w_{\mathfrak{s}}} = \overline{w_{\mathfrak{t}}} \\ 0 & \text{otherwise.} \end{cases} \tag{110}$$

PROOF. By expanding the left-hand side of (110) we get

$$m_{\mathbb{t}^\Lambda \mathbb{s}} m_{\mathbb{t}\mathbb{t}^{\overline{\Lambda}}} = \mathbb{E}_{A_\Lambda} x_{\mathbb{t}^\Lambda \mathbb{s}_0} g_{w_{\mathfrak{s}}} g^*_{w_{\mathfrak{t}}} \mathbb{E}_{A_{\overline{\Lambda}}} x_{\mathbb{t}_0 \mathbb{t}^{\overline{\Lambda}}} = x_{\mathbb{t}^\Lambda \mathbb{s}_0} g_{w_{\mathfrak{s}}} \mathbb{E}_{A_\Lambda w_{\mathfrak{s}}} \mathbb{E}_{A_{\overline{\Lambda}} w_{\mathfrak{t}}} g^*_{w_{\mathfrak{t}}} x_{\mathbb{t}_0 \mathbb{t}^{\overline{\Lambda}}} \tag{111}$$

Using that the $\mathbb{E}_A$'s are orthogonal idempotents we get that (111) is nonzero if and only if $\mathbb{E}_{A_\Lambda w_{\mathfrak{s}}} = \mathbb{E}_{A_{\overline{\Lambda}} w_{\mathfrak{t}}} = \mathbb{E}_{A_\Lambda w_{\mathfrak{t}}}$ but this occurs if and only if $B_y w_{\mathfrak{s}} = w_{\mathfrak{t}}$ for some $y \in \mathfrak{S}_\Lambda^k$ and in that case we get (110) directly from (111). □

The orbit set $\{\overline{w_{\mathfrak{s}}}\}$ for $(\mathfrak{s} \mid \mathbf{u})$ running over $\mathcal{L}_n(\alpha)$ is in bijection with the set of set partitions of type α and so has cardinality given by the Faà di Bruno coefficient $b_n(\alpha)$. Recall that for any algebra $\mathcal{A}$ we denote by $\mathrm{Mat}_N(\mathcal{A})$ the algebra of $N \times N$-matrices with entries in $\mathcal{A}$. We now introduce an arbitrary total order on the above orbit set $\{\overline{w_{\mathfrak{s}}}\}$ and denote by $M_{\mathfrak{s}\mathfrak{t}}$ the elementary matrix of $\mathrm{Mat}_{b_n(\alpha)}\big(\mathcal{H}_\alpha^{wr}(q)\big)$ which is equal to 1 at the intersection of the row and column given by $\mathfrak{s}$ and $\mathfrak{t}$, and is zero otherwise. We can now prove our promised isomorphism Theorem.

Theorem 59 Let α be a partition of n. Then the S-linear map ψ_α induced by $m_{\mathbb{s}\mathbb{t}} \mapsto x_{\mathbb{s}_0 \mathbb{t}_0} M_{\mathfrak{s}\mathfrak{t}}$ is an isomorphism of S-algebras $\mathcal{E}_n^\alpha(q) \longrightarrow \mathrm{Mat}_{b_n(\alpha)}\big(\mathcal{H}_\alpha^{wr}(q)\big)$. A similar isomorphism holds for the specialized algebra $\mathcal{E}_n^{\alpha,\mathcal{K}}(q) := \mathcal{E}_n^\alpha(q) \otimes_S \mathcal{K}$ where $\mathcal{K}$ is an S-algebra as above.

PROOF. Since ψ_α maps a basis to a basis, in the S as well as in the $\mathcal{K}$-situation, it is an isomorphism and so we need only show that ψ_α preserves the multiplications. Suppose that $\Lambda, \overline{\Lambda} \in \mathcal{L}_n(\alpha)$. By the Lemma we have for a pair of standard Λ-tableaux $\mathbb{s} = (\mathfrak{s} \mid \mathbf{u}_1)$, $\mathbb{t} = (\mathfrak{t} \mid \mathbf{u}_2)$ and for a pair of standard $\overline{\Lambda}$-tableaux $\mathbb{u} = (\mathfrak{u} \mid \mathbf{u}_3)$, $\mathbb{v} = (\mathfrak{v} \mid \mathbf{u}_4)$ that

$$m_{\mathbb{s}\mathbb{t}}^\Lambda m_{\mathbb{u}\mathbb{v}}^{\overline{\Lambda}} = \begin{cases} g^*_{w_{\mathfrak{s}}} \mathbb{E}_{A_\lambda} x_{\mathbb{s}_0 \mathbb{t}_0} x_{\mathbb{u}_0 \mathbb{v}_0} g_{w_{\mathfrak{v}}} & \text{if } \overline{w_{\mathfrak{t}}} = \overline{w_{\mathfrak{u}}} \\ 0 & \text{otherwise} \end{cases}$$

On the other hand by the matrix product formula $M_{\mathbf{st}} M_{\mathbf{uv}} = \delta_{\mathbf{tu}} M_{\mathbf{sv}}$ we have

$$\psi_\alpha(m_{\mathbb{s}\mathbb{t}}^\Lambda) \psi_\alpha(m_{\mathbb{u}\mathbb{v}}^{\overline{\Lambda}}) = \begin{cases} x_{\mathbb{s}_0 \mathbb{t}_0} x_{\mathbb{u}_0 \mathbb{v}_0} M_{\mathfrak{s}\mathfrak{v}} & \text{if } \overline{w_{\mathfrak{t}}} = \overline{w_{\mathfrak{u}}} \\ 0 & \text{otherwise} \end{cases}$$

Thus we get the equality $\psi_\alpha(m_{\mathbb{s}\mathbb{t}}^\Lambda m_{\mathbb{u}\mathbb{v}}^{\overline{\Lambda}}) = \psi_\alpha(m_{\mathbb{s}\mathbb{t}}^\Lambda) \psi_\alpha(m_{\mathbb{u}\mathbb{v}}^{\overline{\Lambda}})$ by expanding the product $x_{\mathbb{s}_0 \mathbb{t}_0} x_{\mathbb{u}_0 \mathbb{v}_0}$ and then applying directly the definition of ψ_α. □

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